Logarithms

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8 Logarithms

In this chapter, we introduce logarithms.

8.1 Introduction

We are already familiar with exponents, and certain exponential equations. For example, $5^2 = 25$, or $3^3 = 27$. We may ask ourselves, given an exponent k, what is b^k? We know when we see 5^2, we must multiply 5 by itself twice and obtain 25. Using our exponent rules, we are able to simplify and evaluate expressions with exponents.

But now, suppose instead we are given the base and the result, but not the exponent. For example, consider the equation

$$4^x = 16.$$

How do we find x? In this case it is not too hard, as one can brute force there way to the answer $x = 2$. However, as we'll see, we need heavier machinery to eventually handle some more complicated equations.

Here is where *logarithms* come in.

$$y = \log_b(x).$$

This is read as "y equals log base b of x." This expression translates to the following: "y is the power that we must raise b to in order to get x." Or, if we are asked to evaluate something like $\log_b(x)$ we ask ourselves "to what power must we raise b to in order to get x." The keen-eyed student will notice that logarithms are the inverses of exponential functions. Here b is the base, and b is always positive. If b is not written, we assume that $b = 10$.

Let's look at a few examples:

Example – Logarithm Basics
What value of y makes the following equations true?

1. $y = \log_2(16)$
2. $y = \log_2\left(\frac{1}{4}\right)$
3. $y = \log_5(25)$

For (1), we ask ourselves: "to what power must we raise 2 to in order to get 16." In other words, we are solving the equation $2^y = 16$ for y. The solution is 4.
For (2), we ask ourselves: "to what power must we raise 2 to in order to get $\frac{1}{4}$? Again, this is like solving the equation $2^y = \frac{1}{4}$. Here the solution is -2! Notice that we obtained a negative number. This makes sense; $\frac{1}{4}$ is a fraction, so $2^2 = 4$, however we must remember to flip (make the exponent negative to get the fraction.)
For (3), the solution is 2. So we may complete the equations by substituting the values we found for y.

1. $4 = \log_2(16)$
2. $-2 = \log_2\left(\frac{1}{4}\right)$
3. $2 = \log_5(25)$

Often one finds it helpful to convert mathematical expressions into english words when dealing with logarithms. So, in math:

$$b^y = x \iff \log_b(x) = y.$$

On the left "b raised to the y power yields x", and on the right, "y is the power that we waise b to in order to obtain x." Via this reasoning, one can convert between exponential and logarithmic equations without doing algebra. We'll see with logarithm properties that there is another way.

On a final note, notice: while $\log_b(x)$ can evaluate to a negative number like we saw with $\log_2(1/4)$, we cannot have a negative number as an input. That is, $\log_2(-2)$ is **undefined.** There is no exponent that, when given to 2, yields -2. In general, for a positive base b, there is no exponent which makes it negative. In particular, $\log_b(0)$ is undefined; there is no exponent which causes $b^y = 0$.

8.2 Properties of Logarithms

Here we cover the properties of logarithms. There are three:

Name	Property	Example
Product Rule	$\log_b(p) + \log_b(q) = \log_b(p \cdot q)$	$\log_2(4) + \log_2(4) = \log_2(16)$
Quotient Rule	$\log_b(p) - \log_b(q) = \log_b(p/q)$	$\log_2(8) - \log_2(4) = \log_2(2)$
Exponent Rule	$\log_b(p^q) = q \log_b(p)$	$\log_3(27^2) = 2 \log_3(27)$

The reader is encouraged to calculate the values in the examples given in the table to verify each rule.

8.2.1 Converting Between Exponential and Logarithmic Equations

Before we look at examples, let's get back to the point about converting exponential equations to logarithmic equations and vice versa. If $b^y = x$, then we can take the log of both sides:

$$\begin{aligned} b^y &= x \\ \log_b(b^y) &= \log_b(x) \\ y \log_b(b) &= \log_b(x) && \text{(Exponent Rule)} \\ y &= \log_b(x) && (\log_b(b) = 1) \end{aligned}$$

All we needed was the exponent rule and the fact that $\log_b(b) = 1$ for any b; the power that we must raise b to in order to get b itself is 1. Similarly, if we want to convert $y = \log_b(x)$ into an exponential equation, we make each side an exponent of b and solve.

$$\begin{aligned} y &= \log_b(x) \\ b^y &= b^{\log_b(x)} \\ b^y &= x && (b^{\log_b x} = x) \end{aligned}$$

Here, we relied on the fact that for any base b we have $b^{\log_b(x)} = x$. Loosely, the b and $\log_b$ cancel; however what is actually going here is $\log_b(x)$ is the exponent we raise b to in order to get x. So the expression $b^{\log_b(x)}$ is a bit circular; we obtain the exponent we raise b to in order to get x, then directly raise b to that exponent.

Let us do some examples.

Example – Converting Equations

1. If $\log_{10}(y) = 2$, what does y equal?

   Here, we can solve using algebra, or english. In algebra, we make each side an exponent of 10 since our base is 10:

   $$\begin{aligned} \log_{10}(y) &= 2 \\ 10^{\log_{10}(y)} &= 10^2 \\ y &= 10^2 && (b^{\log_b(\blacksquare)} = \blacksquare) \\ y &= 100 \end{aligned}$$

   In english the equation reads "the power we raise 10 to in order to obtain y is 2", allowing us to write $y = 10^2$.

2. Convert $3^4 = 81$ to a logarithmic equation.

   We have a base of 3, so let's use $\log_3$.

   $$\begin{aligned} 3^4 &= 81 \\ \log_3(3^4) &= \log_3(81) \\ 4\log_3(3) &= \log_3(81) && \text{(Exponent Rule)} \\ 4 &= \log_3(81) && (\log_3(3) = 1) \end{aligned}$$

3. Convert $4^a = 12$ to a logarithmic equation.

   Here our base is 4.

   $$\begin{aligned} 4^a &= 12 \\ \log_4(4^a) &= \log_4(12) \\ a\log_4(4) &= \log_4(12) && \text{(Exponent Rule)} \\ a &= \log_4(12) && (\log_b(b) = 1) \end{aligned}$$

Whether translating through english, or manipulating using algebra, with enough practice one should be comfortable solving exponential and logarithmic equations of these kinds. So far we've only scratched the surface to get a better understanding of what logarithms are, and how we can use them to solve equations involving unknown variables as exponents (an x in the exponent.)

8.2.2 Expanding and Condensing Logarithmic Expressions

Similar to learning exponent rules, we must practice expanding and condensing logarithmic expressions using our rules. Just like exponent rules, there are many ways to combine and expand to get to a final result.

Expanding Logarithms

1. Write the expanded form of $\log\left(\frac{3}{5}x^2\right)$.

$$\begin{aligned}
\log\left(\frac{3}{5}x^2\right) &= \log(3/5) + \log(x^2) && \text{(Product Rule)}\\
&= \log(3) - \log(5) + \log(x^2) && \text{(Quotient Rule)}\\
&= \log(3) - \log(5) + 2\log(x) && \text{(Exponent Rule)}
\end{aligned}$$

2. Write the expanded form of $\log\left(\frac{y^4}{\sqrt[3]{xz^2}}\right)$

   This one looks difficult, but if we know our exponent rules and keep our head straight, it is a step by step process.

$$\begin{aligned}
\log\left(\frac{y^4}{\sqrt[3]{xz^2}}\right) &= \log(y^4) - \log(\sqrt[3]{xz^2} && \text{(Quotient Rule)}\\
&= \log(y^4) - \log((xz^2)^{1/3}) && (\sqrt[k]{\blacksquare} = \blacksquare^{1/k})\\
&= \log(y^4) - \log(x^{1/3}z^{2/3}) && \text{(Distribute Exponent)}\\
&= \log(y^4) - \log(x^{1/3}) + \log(z^{2/3}) && \text{(Product Rule)}\\
&= 4\log(y) - \frac{1}{3}\log(x) + \frac{2}{3}\log(z) && \text{(Exponent Rule)}
\end{aligned}$$

As practice for condensing, take the end result of the previous examples and try to condense them back to get the original equations. As a practice problem, show that

$$2\log(z) + 3(\log(y) - 2\log(x)) = \log\left(\frac{z^2y^3}{x^6}\right).$$

8.3 Solving Exponential and Logarithmic Equations

In this section we will see how we can use the fact that logarithms and exponentials are inverses of each other to solve special equations. Early we saw, for example, when we have a quadratic equation we can take the square root of two sides to solve the equation:

$$\begin{aligned}(x+5)^2 &= 25\\ \sqrt{(x+5)^2} &= \sqrt{25}\\ x+5 &= 5\\ &\vdots\end{aligned}$$

and similarly when solving a radical equation, we square both sides

$$\begin{aligned}\sqrt{x+4} &= 3\\ (\sqrt{x+4})^2 &= (3)^2\\ x+4 &= 9\\ &\vdots\end{aligned}$$

In the case of radical equations, recall that checking our solutions is mandatory. We will see similar behavior with exponentials and logarithms.

The idea with solving exponential equations (equations where unknown variables are in the exponent) is to use the properties of logarithms, in particular the exponent property, to isolate the unknown variable.

Example – Solving Exponential Equations Using Logarithms

1. $5e^{x+2} = 2$

   Here we have e, known as *Euler's constant.* The number e is an irrational number, approximately 2.7. Like many other special constants, we assign it a special letter. In this equation, let us use $\log_e(x) = \ln(x)$, the natural log.

   $$\begin{aligned}5e^{x+2} &= 15 & \\ e^{x+2} &= 3 & \text{(Isolate the exponential)}\\ \ln(e^{x+2}) &= \ln(3) & \\ (x+2)\ln(e) &= \ln(3) & \\ x+2 &= \ln(3) & (\ln(e)=1)\\ x &= \ln(3)-2 &\end{aligned}$$

   We cannot simplify any longer, we leave our solution in **exact form.**

2. Solve for y: $5^{y+3} = 6$.

   Here we have a single exponential term with a base of 5. So let's "take the log of both sides", with the base of our logarithm as 5.

$$\begin{aligned} 5^{y+3} &= 6 \\ \log_5(5^{y+3}) &= \log_5(6) \\ (y+3)\log_5(5) &= \log_5(6) && \text{(Exponent Rule)} \\ y+3 &= \log_5(6) && (\log_b(b) = 1) \\ y &= \log_5(6) - 3 && (\log_b(b) = 1) \end{aligned}$$

   We could have used log with any base. The solution we obtained, $y = \log_5(6) - 3$, is in **exact form**. We simplified it as best we can.

3. Solve for x: $12^{4x} = 5^{-x+3}$

   Let's use the natural log, ln here, since we have two different bases and we can't be sure things will simplify nicely.

$$\begin{aligned} 12^{4x} &= 5^{-x+3} \\ \ln(12^{4x}) &= \ln(5^{-x+3}) \\ 4x\ln(12) &= (-x+3)\ln(5) && \text{(Exponent Rule)} \\ 4x\ln(12) &= -x\ln(5) + 3\ln(5) && \text{(Distribute)} \\ 4x\ln(12) + x\ln(5) &= 3\ln(5) \\ x(4\ln(12) + \ln(5)) &= 3\ln(5) && \text{(Factor } x\text{)} \\ x &= \frac{3\ln(5)}{4\ln(12) + \ln(5)} \end{aligned}$$

Now let's look at logarithmic equations.

Example – Solving Logarithmic Equations

1. Solve for x: $\log_5(-2x + 10) = 2$.

   In equations with logs, we can make both sides of the equation an exponent of a particular base. Here, the base of the log term is 5. This comes from the fact that if $x = y$, then $b^x = b^y$.

$$
\begin{aligned}
\log_5(-2x+10) &= 2 \\
5^{\log_5(-2x+10)} &= 5^2 \\
-2x+10 &= 25 && (b^{\log_b(x)} = x) \\
-2x &= 15 \\
x &= -15/2 \text{ or } -7.5
\end{aligned}
$$

Plugging in to check our work:

$$
\begin{aligned}
\log_5(-2(-7.5)+10) &= 2 \\
\log_5(15+10) &= 2 \\
\log_5(25) &= 2 \\
2 &= 2 \checkmark
\end{aligned}
$$

2. Solve for x: $\log_5(x+4) = 1 - \log_5(x+8)$.

   Now we have a logarithmic equation with two logs, but they each have the same base. We use our logarithm properties to solve the equation; at each step there is only one logical step forward.

$$
\begin{aligned}
\log_5(x+4) &= 1 - \log_5(x+8) \\
\log_5(x+4) + \log_5(x+8) &= 1 \\
\log_5((x+4)(x+8)) &= 1 && \text{(Product Rule)} \\
5^{\log_5((x+4)(x+8))} &= 5^1 \\
(x+4)(x+8) &= 5 \\
x^2 + 12x + 32 &= 5 \\
x^2 + 12x + 27 &= 0 \\
(x+3)(x+9) &= 0
\end{aligned}
$$

   Notice what we've done here. We combined the log terms into one term by using the Product Rule, which reduced the problem to something we already know how to solve (a logarithmic equation with one logarithm.) Then, we reduce the equation further to a quadratic equation. The two possible solutions are $x = -3$ and $x = -9$. Let's check our solutions.

   Check $x = -3$.

$$
\begin{aligned}
\log_5(x+4) &= 1 - \log_5(x+8) \\
\log_5((-3)+4) &= 1 - \log_5((-3)+8) \\
\log_5(1) &= 1 - \log_5(5) \\
0 &= 0 \checkmark
\end{aligned}
$$

   Check $x = -9$.

$$
\begin{aligned}
\log_5((-9)+4) &= 1 - \log_5((-9)+8) \\
\log_5(-5) &= 1 - \log_5(-1) \text{ ✗}
\end{aligned}
$$

Notice that $x = -9$ is not a solution! **It is mandatory to check solutions when working with logarithmic equations.** In particular, we must check solutions when we use the product rule to combine logs, as we may introduce **extraneous solutions.**

8.4 The Logarithmic Graph

Now we consider the function $f(x) = \log_b(x)$. As we've mentioned previously, the log function $f(x) = \log_b(x)$ is the inverse function of $g(x) = b^x$; while exponentials take exponents as inputs, logarithms take numbers as inputs and produce the exponent that is needed to make the base equal to the number. Again, in math,

$$y = b^x \iff \log_b(y) = x.$$

Graphically, the inverse of a function is the reflection over the line $y = x$.

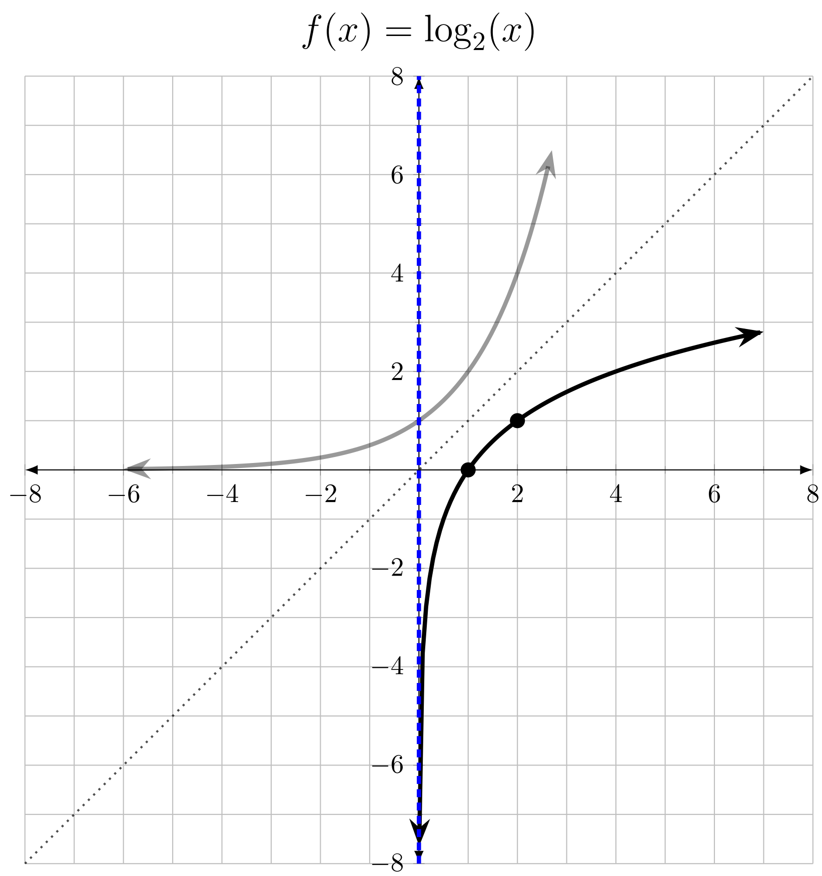


Figure 1: The function $f(x) = \log_2(x)$ is depicted in bold, with the inverse $f^{-1}(x) = 2^x$ in gray.

Notice the base function characteristics:

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Vertical asymptote at $x = 0$.
- x-int $(1, 0)$

Knowing that the logarithmic functions are inverses of exponential functions, compare the characteristics of $f(x) = \log_b(x)$ and $f^{-1}(x) = b^x$. Notice how the domain and range are swapped, among other things.

8.4.1 Transformations

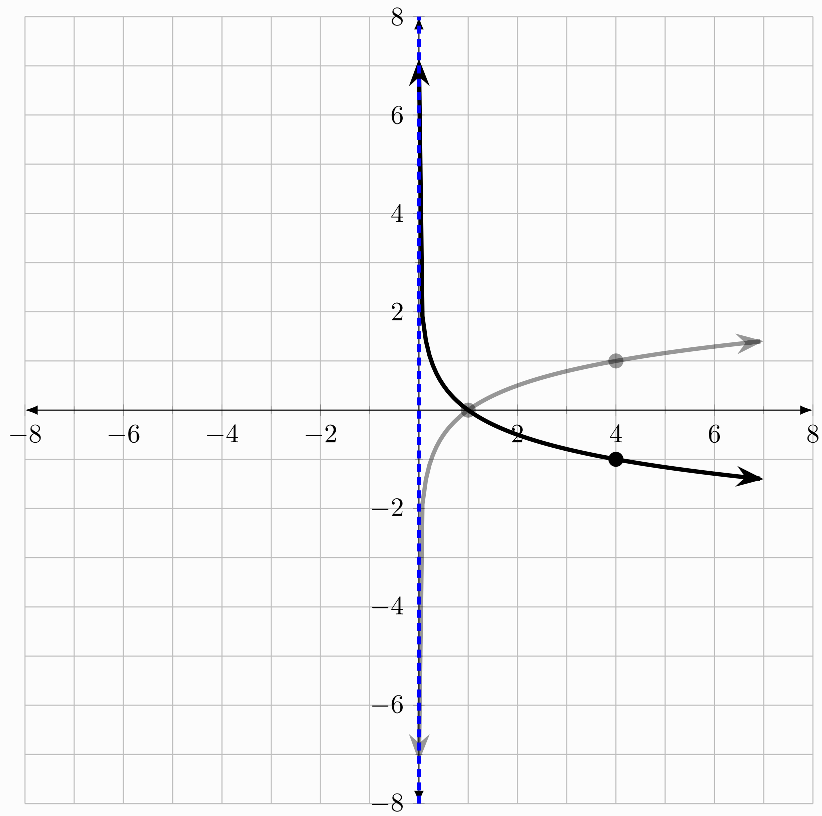
Just as we've seen with exponentials, we can shift horizontally, vertically, and reflect. We have

$$f(x) = a \log_b(x - h) + k.$$

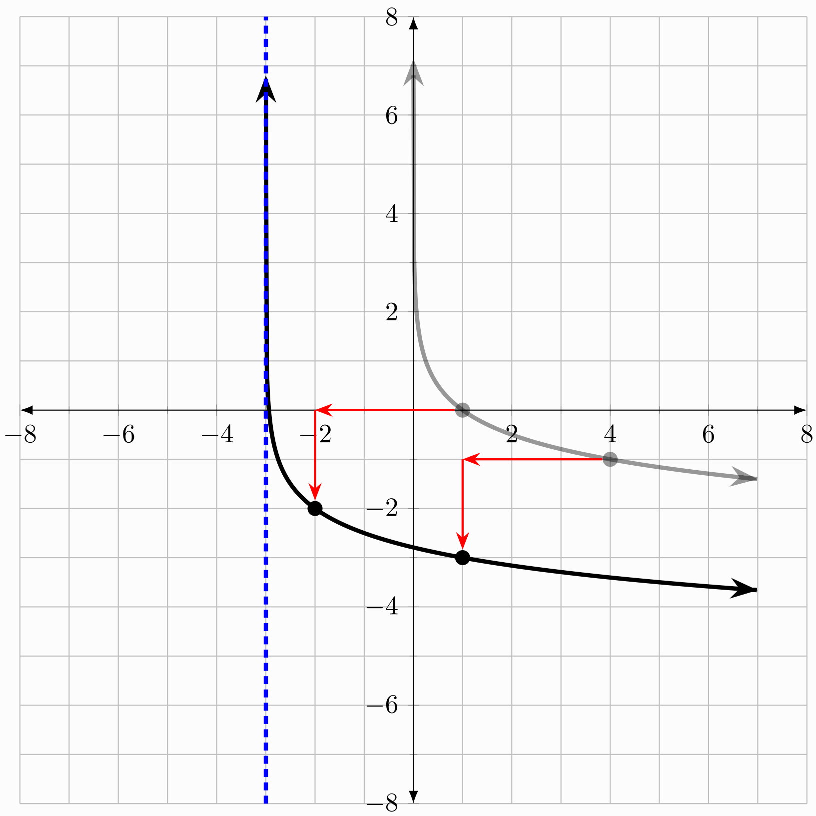
Example – Translations of a Logarithm.
Consider the function $f(x) = -\log_4(x+3) - 2$. Knowing our function translations, we have $a = -1$ which is negative, $h = -3$, and $k = -2$. In other words

- $a < 0$: $f(x)$ is $\log_4(x)$ reflected over a horizontal axis.
- $h = -3$: $f(x)$ is $\log_4(x)$ shifted 3 units to the left.
- $k = -2$: $f(x)$ is $\log_4(x)$ shifted 2 units downward.

So first let us apply the reflection.



Notice the domain, range, vertical asymptote, and x-int have not changed. This will not be the case when we shift!



Notice now that the vertical asymptote shifted 3 units to the left. The domain as then changed from $(0, \infty)$ to $(-3, \infty)$ (just like vertical shifts change the range of the inverse exponential functions.) The range remains unchanged $(-\infty, \infty)$. We've also gained a y-intercept which we did not have before.

As we can see, function translations affect the characteristics of log functions.
For $f(x) = a \log_b(x - h) + k$

- Domain: Solve for when $x - h > 0$. This gives when $x > h$. So (h, ∞).
- Range: $(-\infty, \infty)$.
- Vertical Asymptote at $x = h$.

8.4.2 Finding the Domain

We will now come back to something familiar; finding the domain of a composition of functions. We know that $\log_b(x)$ can only take positive inputs. That is, whenever we have a log term $\log_b(\blacksquare)$, the inside $\blacksquare$ must be strictly greater than 0. In math, $\blacksquare > 0$. Some times we can tell what the domain is from understanding function translations, but there is an algebraic way to solve for the domain.

Finding the Domain of a Logarithmic Function

1. Let $f(x) = \log(-x+5)$.

   The domain of $\log(\blacksquare)$ is all the values satisfying the inequality $\blacksquare > 0$. So we solve:

$$\begin{aligned} -x+5 &> 0 \\ -x &> -5 \\ x &< 5 && \text{(Multiply by } -1\text{, flip sign)} \end{aligned}$$

   So the domain of $f(x)$ is all x values such that $x < 5$. In interval notation, this is $(-\infty, 5)$. Check to see what happens when we plug numbers outside of this interval into f. What happens?

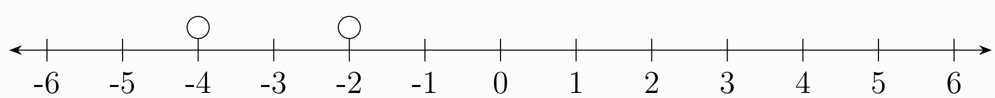
---

2. Let $g(x) = \log(x^2+6x+8)$.

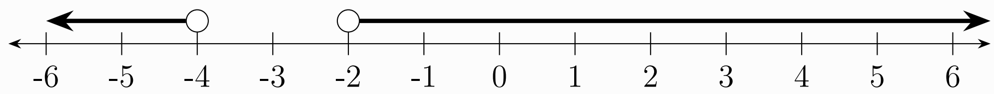
   Again, we need the inside to be positive. So we need to solve $x^2+6x+8 > 0$. First we factor:

$$\begin{aligned} x^2+6x+8 &> 0 \\ (x+2)(x+4) &> 0 \end{aligned}$$

   Now we have a quadratic inequality, or in particular we technically have a rational inequality (with denominator 1). So we set $(x+2)(x+4) = 0$ to obtain critical values, then we find test values. Our critical values are $x = -2$ and $x = -4$, both having open circles on our number line.



Interval	Test Value	Plug In	T/F
$(-\infty, -4)$	-5	$((-5)+2)((-5)+4) = (-3)(-1) = 3 > 0$	True
$(-4, -2)$	-3	$((-3)+2)((-3)+4) = (-1)(1) = -1 \not> 0$	False
$(-2, \infty)$	0	((0)+2)((0)+4) = (2)(4) = 8 ¿ 0	True



3. Let $f(x) = \log\left(\frac{(x-1)}{(x+2)(x-4)}\right)$

   Here we set the inside > 0. That is, we solve the inequality

   $$\frac{x-1}{(x+2)(x-4)} > 0.$$

   The reader is encouraged to review rational inequalities and solve for the domain. One should obtain, in interval notation, $(-2,1) \cup (4,\infty)$.