Exponentials
Summer 2023

Rayan Ibrahim

Math 151 - 901

Contents

7 Exponential Functions

In this chapter we will learn about exponential functions.

7.1 Introduction

Consider the following two functions:

$$f(x) = 2^x$$
$$g(x) = \left(\frac{1}{2}\right)^x$$

Let's make a table of some values. The reader is encouraged to calculate these values on their own.

x	$f(x)$	$g(x)$
-2	$1/4$	4
-1	$1/2$	2
0	1	1
1	2	$1/2$
2	4	$1/4$

These are *exponential functions.* There is a base b and the input x is the exponent:

$$h(x) = b^x.$$

Their graphs are given in Figure 1.

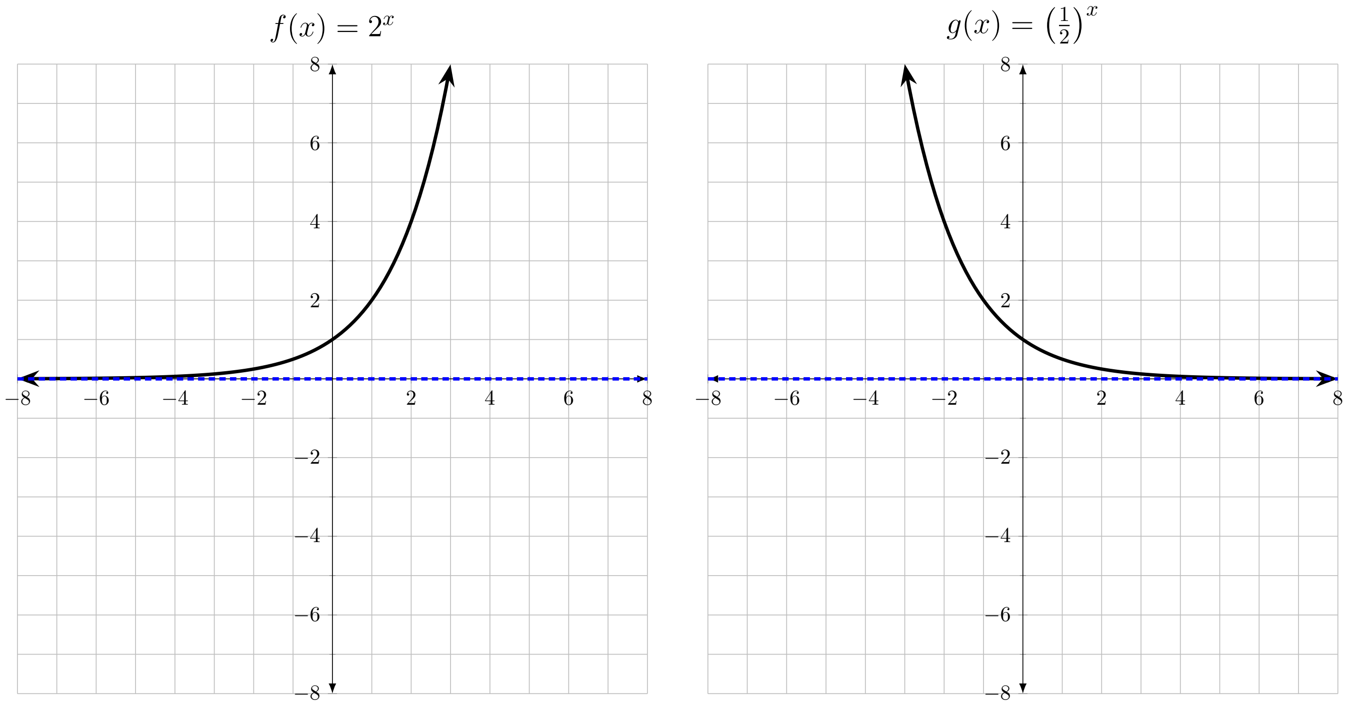


Figure 1: Examples of exponential functions in graph form. Notice that each function has the same y-intercept and horizontal asymptote. Also notice some important values, at $x = 0$ and $x = 1$.

If b is not a proper fraction and positive, then we have exponential growth; think of setting the base to be any whole number. As $x \to \infty$, the function $f(x) \to \infty$, and as $x \to -\infty$, the function $f(x) \to 0$. Notice, if x is negative the base "becomes a fraction", for example

$$f(-2) = 2^{-2} = (2^{-1})^2 = \left(\frac{1}{2}\right)^2.$$

On the other hand, when b is a proper fraction, like $1/2$, as $x \to \infty$, the exponential function goes to 0. This makes sense; in a proper fraction the denominator is larger than the numerator, and when we raise the fraction to an exponent, the denominator will "beat" the numerator (grow faster). This leads to exponential *decay.* Meanwhile, as $x \to -\infty$, the function goes to ∞. Notice that if we have a proper fraction as a base, and we plug in a negative number, the fraction "flips". For example,

$$g(-2) = \left(\frac{1}{2}\right)^{-2} = \left(\left(\frac{1}{2}\right)^{-1}\right)^2 = (2)^2.$$

So as $x \to -\infty$, the function goes to ∞.

So we have exponential decay when $0 < b < 1$ and exponential growth when $b > 1$. Notice, we do not set b to be negative or equal to 1.

Then for any exponential function of the form $f(x) = b^x$, we have

- Domain: $(-\infty, \infty)$. We can plug any number in as an exponent.
- Range: $(0, \infty)$.
- Horizontal asymptote at $x = 0$.
- y-intercept of $(0, 1)$

Notice that these exponential functions do not have any x-intercepts. Many of these characteristics will change when we apply function transformations, but the core behaviors will be very similar.

7.2 Transformations

As we know from our prerequisite knowledge, we can tranform functions; we can shift their graphs left, right, up, or down some number of units. We can stretch and compress functions, and we can reflect. Using our knowledge, we can write a general form for exponential functions as

$$f(x) = ab^{x-h} + k.$$

In this form, $f(x)$ is b^x

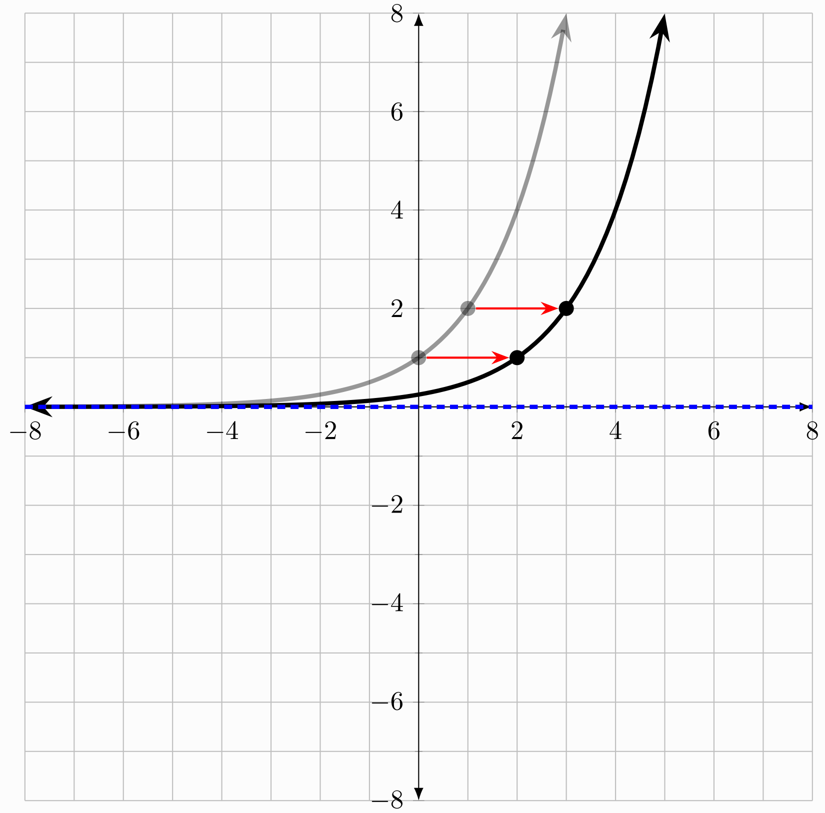
- shifted h units to the left if h is negative and to the right if h is positive.
- shifted k units upward if k is positive and downward if k is negative.
- reflected about a horizontal axis (the asymptote) if a is negative.

These are the important details we focus on. Let us see some examples.

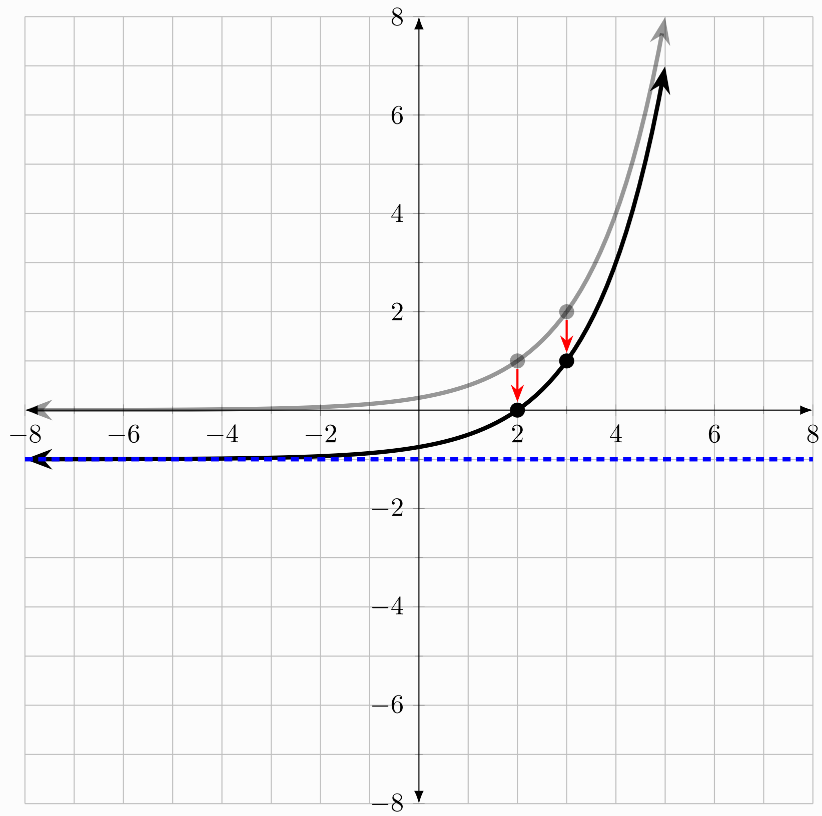
Example – Exponential Function Shifted

Let $f(x) = 2^{x-2} - 1$.

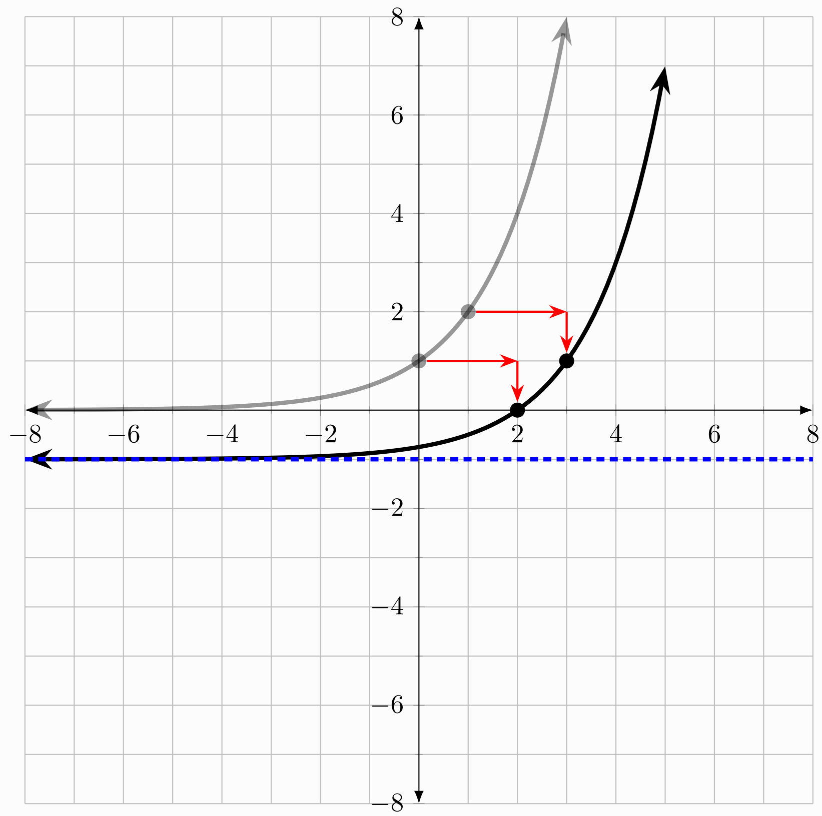
The function given to us is 2^x shifted to the right 2 units and down 1 unit. So we start with $y = 2^x$ and shift to the right 2 units.



Now we shift $y = 2^{x-2}$ one unit down.



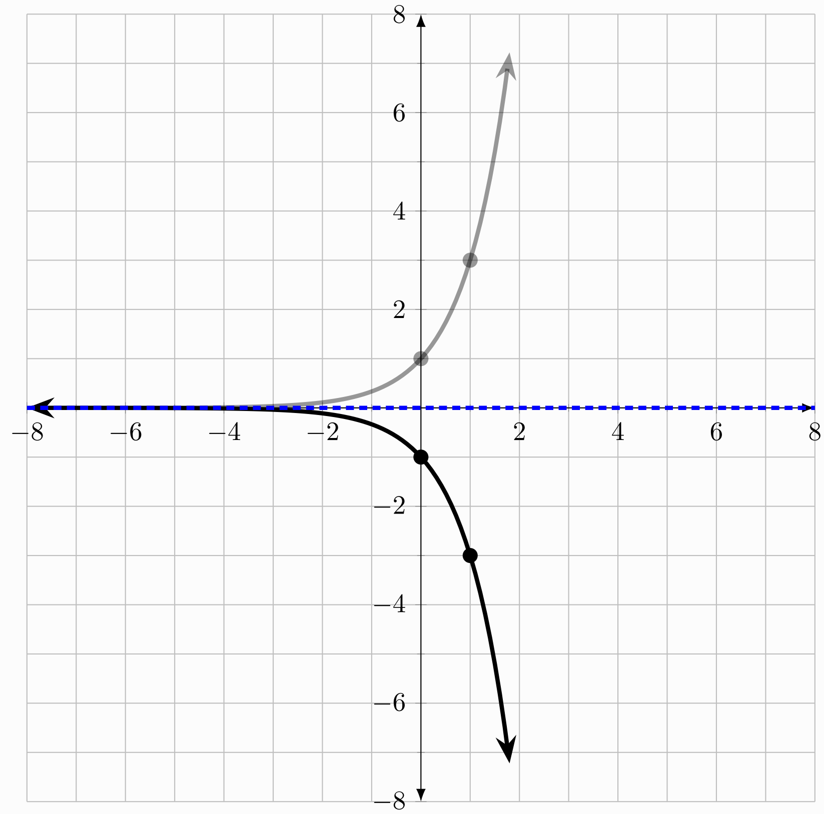
Both shifts at once:



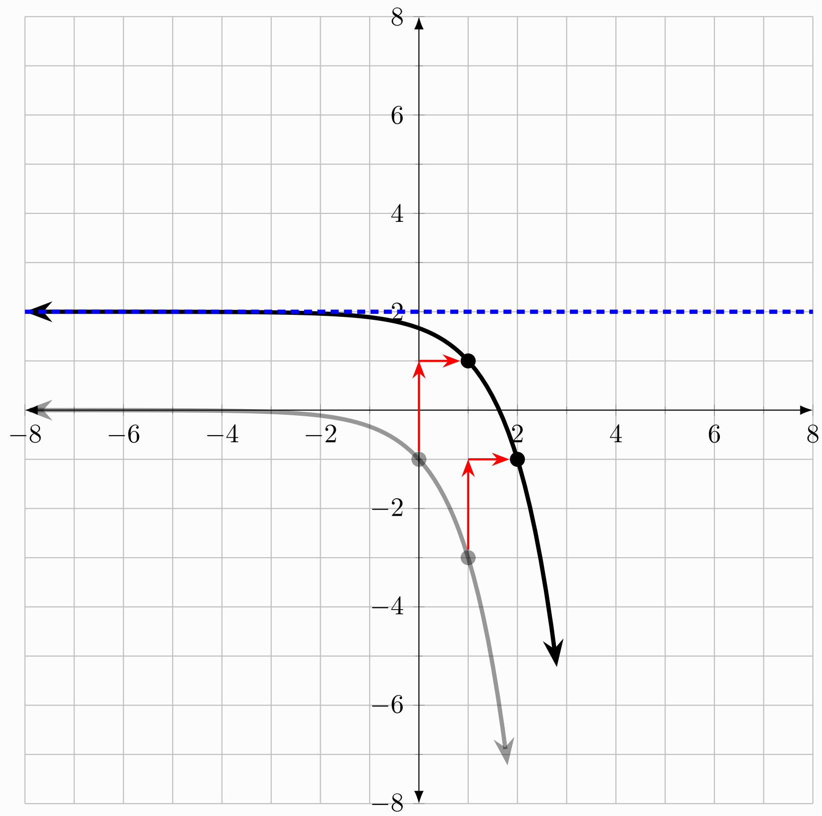
Notice that this new function has several characteristics that our basic exponential functions did not have, or are different.
By shifting downward, the horizontal asymptote goes from $y = 0$ to $y = -1$. Additionally, there is now an x-intercept $(2, 0)$, and our y-intercept is $(0, -3/4)$. The domain is $(-\infty, \infty)$ and the range is $(-1, \infty)$.

Let's do another example, this time with a reflection.

Example – Exponential with Reflection
Let $f(x) = -(3)^{x-1} + 2$.
Here, a is negative. So we have reflection, a shift 1 unit to the right, and a shift 2 units upward.
Let us start with $y = 3^x$ and reflect.



Then we shift as usual.



Notice here that the domain is still $(-\infty, \infty)$. The horizontal asymptote has shifted up 2 units, and is now $y = 2$. The range is $(-\infty, 2)$. There is a y-intercept, $(0, 5/3)$. There is an x-intercept, and we can find it by setting $-(3)^{x-1} + 2 = 0$. However, we will need logarithms to solve these kinds of equations; exponential equations.

So in general for an exponential function

$$f(x) = ab^{x-h} + k$$

- The domain is $(-\infty, \infty)$.
- The range is:
  - $(-\infty, k)$ if a is negative.
  - (k, ∞) if a is positive.
- $y = k$ is the horizontal asymptote.

Summary

List of things you need to know.

- Translations of exponential functions and how they affect the characteristics.