Inverses
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Rayan Ibrahim

Math 151 - 901

Contents

6 Inverse Functions

Recall: A *function* is a rule which defines a relationship between an input variable, say x, and an **unique** output (often referred to as y). Functions can be given names, commonly f, g, or h. When we define a function, we may write something like

$$f(x) = x + 5$$

which is read "f of x equals x plus 5." This function, whose name is f and input is x, takes the input and adds 5 to it, giving us an output. To evaluate a function at a specified value of x, say 5, we write

$$f(5) = (5) + 5 = 10.$$

Everywhere we see an x, we substitute a 5, then we simplify. Again, the output is **unique**; when we plug in a number for x, the function will output exactly one number. To test this on a graph, we use the vertical line test. If we can draw a vertical line on top of a graph, then that signifies that there are two y values which share the same x value; the given graph is not that of a function.

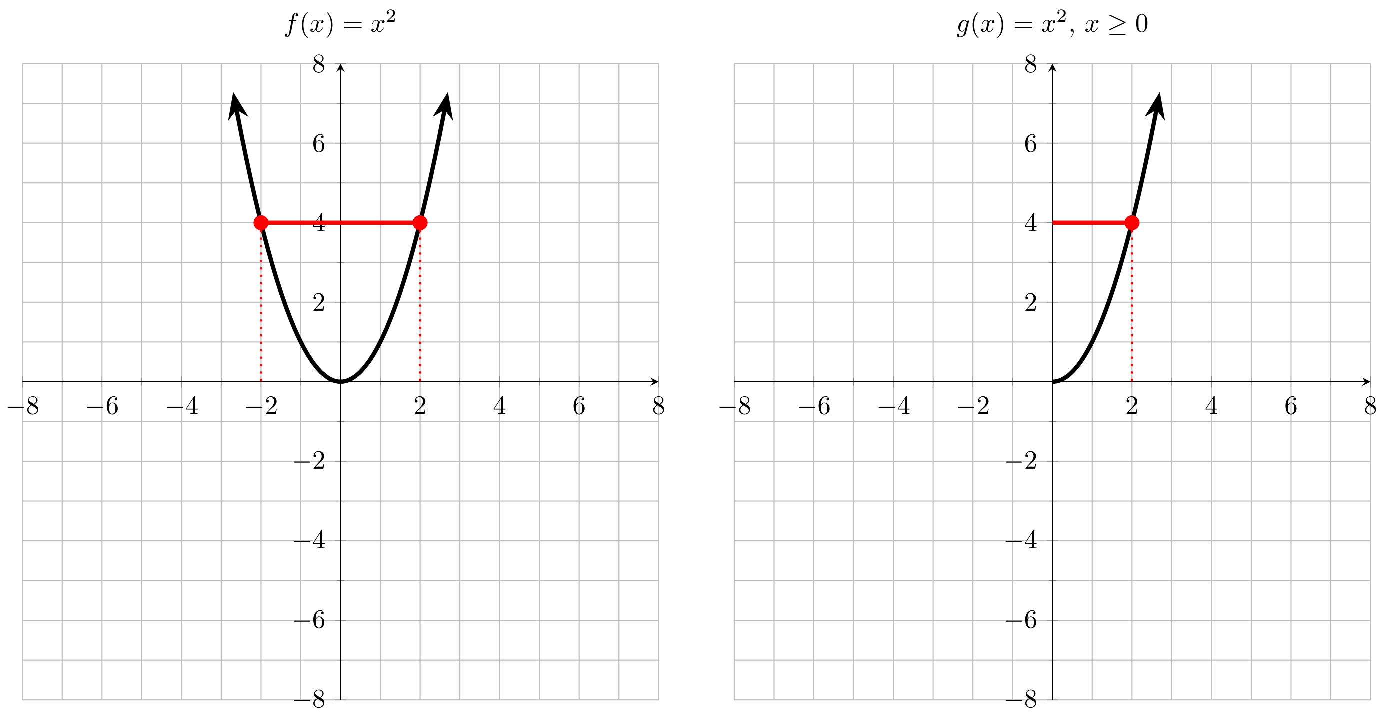
Now we define what it means for a function to be one-to-one.

Definition – One-to-One

A function is said to be **one-to-one** if for every output there is exactly one input.

To test if a given graph is that of a one-to-one function, we use the horizontal line test; if we can draw a horizontal line on a graph where it crosses the graph twice, then there is an output value (a value on the y-axis) which comes from two different x values.

It is easy to get confused about what it means to be a function, and what it means to be a one-to-one function. A function takes an input and gives **exactly one output.** A function is one-to-one if every y-value comes from a unique x value. For example, let us look at the function $f(x) = x^2$.



Certainly $f(x) = x^2$ is a function. We plug in a number, we get exactly one number out. However, $f(x) = x^2$ is not one-to-one. The output 4 comes from two different x values. That is, $f(-2) = f(2) = 4$. We can see that x^2 horrifically fails the horizontal line test. However, if we are selective of the domain of our function, then we can make a one-to-one function. In the case of x^2, if we only consider the portion where $x \geq 0$, then we have a one-to-one function. In general, we can take "one half" of any quadratic function, as they are all symmetric about the vertex.

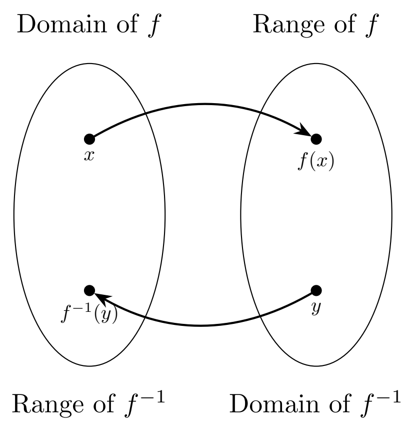
The concept of a one-to-one function comes in handy when we want to define an *inverse* for our function.

Definition – Inverse

Let f be a one-to-one function. Then f has an inverse function, denoted f^{-1} such that if $f(x) = y$, then $f^{-1}(y) = x$.

In words, while f takes inputs and gives outputs, f^{-1} takes outputs of f and returns the corresponding input. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

One should reason with themselves to truly understand the definition; why is it that the compositions $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.



Now we may understand why the concept of one-to-one is important. The function $f(x) = x^2$ is not one-to-one; it does not have an inverse. If it had an inverse f^{-1}, then $f^{-1}(4)$ would need to output both -2 and 2, and so f^{-1} would not be a function. However, if we specify that $f(x) = x^2,\ x \geq 0$, then f does indeed have an inverse $f^{-1}(x) = \sqrt{x}$.

6.1 Finding the Inverse

Again, the inverse takes y values and returns x values. That is, for every point (x, y) in the graph of f, we have the point (y, x) in the graph of f^{-1}.

Example – Given Coordinates of a Function

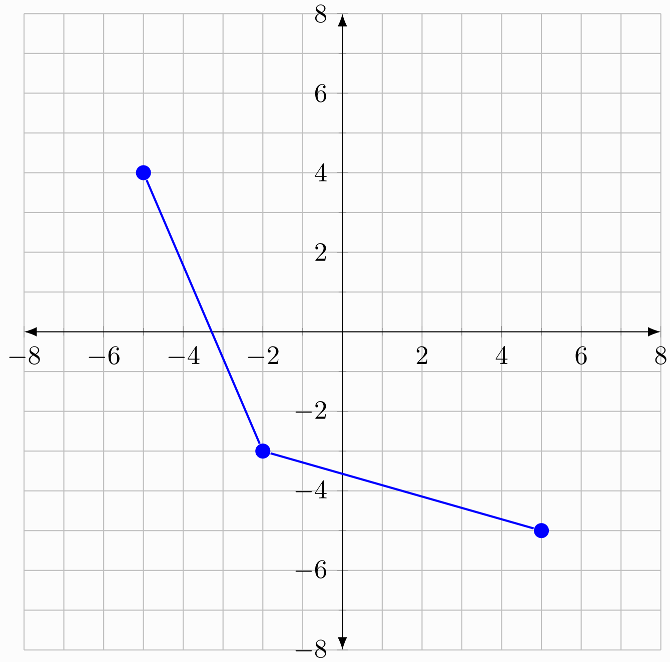
Let $g = \{(-8, 0), (-4, 5), (2, 3), (3, 4), (7, -4)\}$

Notice that g is a function; each x value comes with exactly one y value. We ask ourselves, is g one-to-one? We check, does each y value have its own, unique x

value. Indeed, each y value appears once and only once. Now we want to evaluate, say, $g^{-1}(3)$. Two coordinates have 3 in them, $(2,3)$ and $(3,4)$. However, we are evaluating $g^{-1}(3)$, and the inverse takes y values as inputs. In words, $g^{-1}(3)$ asks: "what value of x, when plugged into g, gives 3?" We see then that $g^{-1}(3) = 2$, since $g(2) = 3$.
With similar reasoning, we see that $g^{-1}(-4) = 7$ or $g^{-1}(0) = -8$.

A list of coordinates is the first step. One half step and we begin to examine graphs.

Example – Inverse Given a Graph
Consider the following function:



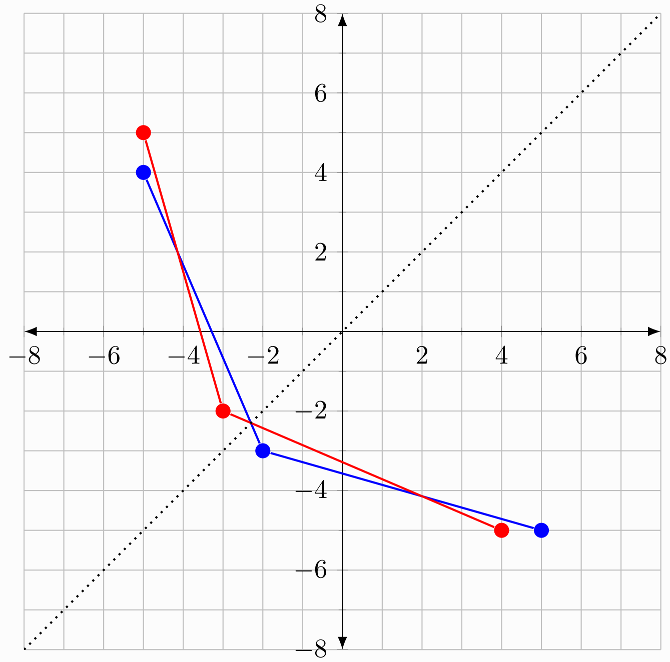
The function depicted has domain $[-5, 5]$ and range $[-5, 4]$. We make a table of the (x, y) values:

x	$f(x)$
-5	4
-2	-3
5	-5

The inverse, $f^{-1}(x)$ takes y values and gives the x values; so we switch the coordinates.

x	$f^{-1}(x)$
4	-5
-3	-2
-5	5

Now we can plot our points, and connect them:



There is some geometric meaning here. The graph of a function, when reflected over the line $y = x$, yields the graph of the inverse. This is what is happening when we switch the x and y coordinates.

In general, some times we are not given such a nice graph. However, one can pick a few points on a sketch of a graph, swap the x and y coordinates, and sketch the inverse. Then, one checks their work by drawing $y = x$ and checking if the sketches are indeed reflections of each other.

Suppose instead of a set of coordinates or a graph, we are given an equation. To find the inverse of a function $f(x) = \blacksquare$ we

1. Write $y = \blacksquare$.

2. Replace every instance of x with y, and every instance of y with x. We get $x = \blacksquare$ where every x in $\blacksquare$ is replaced with a y.

3. Solve for y in terms of x; isolate y with algebra. We will get a new equation of the form $y =$.

4. Replace y with $f^{-1}(x)$.

Example – Finding the Inverse From Equations
Let $f(x) = \frac{4x+1}{7}$. Find f^{-1}.

We are given a line, which can be written as $f(x) = \frac{4}{7}x + \frac{1}{7}$. Following our procedure:

$$\begin{aligned}
y &= \frac{4x+1}{7} && (f(x) = y) \\
x &= \frac{4y+1}{7} && (y = x) \\
7x &= 4y + 1 \\
7x - 1 &= 4y \\
\frac{7x-1}{4} &= y && (\text{Solve for } y) \\
\frac{7x-1}{4} &= f^{-1}(x)
\end{aligned}$$

We can check our answer with some specific values. Let $x = 5$. Then $f(2) = \frac{4(5)+1}{7} = \frac{21}{7} = 3$. Now when we evaluate $f^{-1}(3)$ we should get 2. Indeed:

$$\begin{aligned}
f^{-1}(3) &= \frac{7(3)-1}{4} \\
&= \frac{20}{4} \\
&= 5.
\end{aligned}$$

The reader is encouraged to check other values of x.

Let us do another example with some more involved algebra.

Example – Finding the Inverse (Harder)
Let $f(x) = \frac{3x}{5x+1}$. Find $f^{-1}(x)$.

$$
\begin{aligned}
y &= \frac{3x}{5x+1} \\
x &= \frac{3y}{5y+1} \\
x(5y+1) &= 3y && \text{(Clear Denominator)} \\
5xy + x &= 3y && \text{(Distribute } x\text{)} \\
5xy - 3y &= -x && \text{(Move } 3y \text{ left and } x \text{ right.)} \\
y(5x-3) &= -x && \text{(Factor } y\text{)} \\
y &= \frac{-x}{5x-3} && \text{(Solve for } y\text{)} \\
f^{-1}(x) &= \frac{-x}{5x-3}
\end{aligned}
$$

Summary

List of things you need to know.

- A one-to-one function is a function where each output has a unique input.
  - Use the horizontal line test on a graph to test if a function is one-to-one
- The inverse f^{-1} of a function f takes outputs $y = f(x)$ and produces inputs $x = f^{-1}(y)$.
- A function has an inverse if and only if it is one-to-one.
- The domain and range of an inverse is the range and domain of the function, respectively.
- To find the inverse of a function from its equation, write the equation in the form $y =$, change every x to y and every y to x, then solve for y.