Compositions
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Contents

5 Composition of Functions

In this chapter we will learn about composition of functions.

Given two functions, we may *compose* them; one function may act as an input of another. Let $f(x)$ and $g(x)$ be two functions.

We write a composition as $f(g(x))$ or $g(f(x))$, read as "f of g of x" and "g of f of x" respectively. An alternate notation of $f(g(x))$ is $(f \circ g)(x)$, which can be read as "f composed with g evaluated at x." T

To evaluate the composition $f(g(x)) = (f \circ g)(x)$, we first evaluate $g(x)$, then we plug the result in to $f(x)$. Note, it is not always the case that $f(g(x)) = g(f(x))$. In other words, **the order of composition matters**. To get the notation straight:

$$(f \circ g)(x) = f(g(x))$$
$$(g \circ f)(x) = g(f(x))$$

5.1 Evaluating Compositions

Let's do an example with some concrete functions.

Example – Evaluating Compositions Let $f(x) = x + 5$ and $g(x) = x^2 + 2$. Then

$$f(g(x)) = (x^2 + 2) + 5 = x^2 + 7$$
$$g(f(x)) = (x + 5)^2 + 2 = x^2 + 10x + 27$$

Notice how $f(g(x)) \neq g(f(x))$.
We can compute some values for the compositions. For example, for $f(g(2))$ we compute $g(2) = (2)^2 + 2 = 6$, then plug 6 into f, so

$$f(g(2)) = f(2^2 + 2) = f(6) = (6) + 5 = 11.$$

Or, like above, we can write the composition in its general form, such as $f(g(x)) = x^2 + 7$ and then evaluate. Here are some examples, showing the many ways we can compute a composition of functions.

- $(g \circ f)(2) = ((2) + 5)^2 + 2 = (7)^2 + 2 = 51$
- $g(f(10)) = ((10) + 5)^2 + 2 = (15)^2 + 2 = 227$
- $(f \circ g)(1) = f(g(1)) = f((1)^2 + 2) = f(3) = (3) + 5 = 8$

Try to work out the following on your own. Let $p(x) = \sqrt{x + 1}$ and $q(x) = x^2 + 5$. Show that

- $(p \circ q)(x) = \sqrt{x^2 + 6}$
- $(q \circ p)(x) = x + 6$

We may also evaluate compositions from a table.

Example – Compositions With Tables Consider the following table of values for two functions r and q.

x	$r(x)$	$q(x)$
1	4	12
2	3	10
3	1	5
4	2	14

Compute $q(r(2))$ and $q(r(4))$.

To evaluate these, we first must evaluate the inside. So

- $q(r(2))$: First we evaluate $r(2)$, which from the table we know is 3.

x	$r(x)$	$q(x)$
1	4	12
2	**3**	10
3	1	5
4	2	14

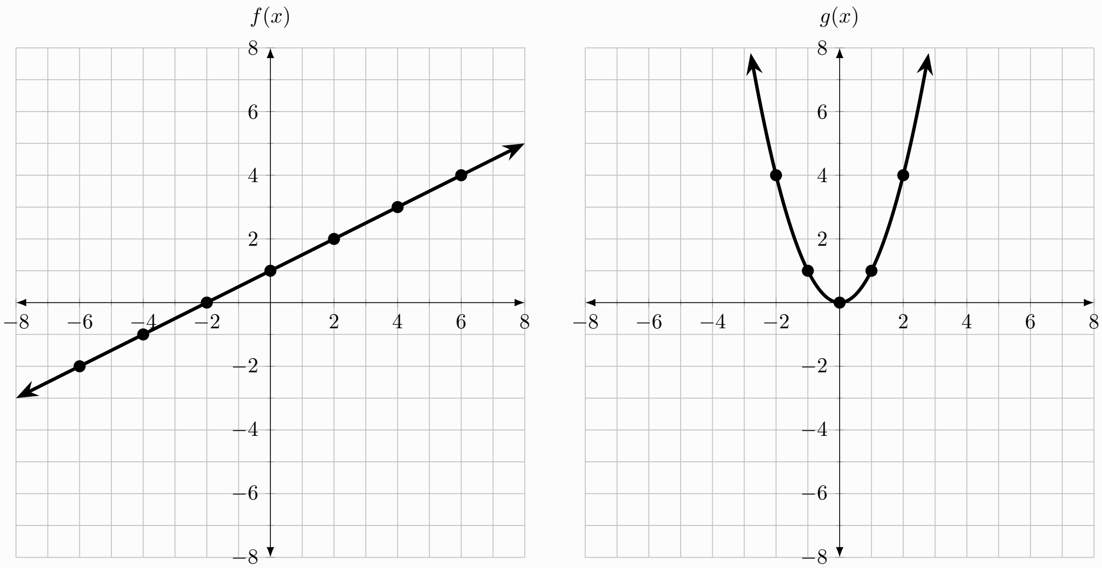
  So $q(r(2)) = q(3)$. Looking at the third row, we know $q(3) = 5$.

x	$r(x)$	$q(x)$
1	4	12
2	3	10
3	1	**5**
4	2	14

- $q(r(4))$: First we evaluate $r(4)$, which from the table we know is 2. So $q(r(4)) = q(2)$. Looking at the third row, we know $q(2) = 10$.

Finally we can evaluate compositions via graphs. Note, it is always the same process. For any composition $(f \circ g)(x) = f(g(x))$, we first evaluate $g(x)$ to get a number, then we plug in that number to f. On a graph this is nothing new; given an x coordinate, we find the y coordinate on a graph of g, then use that y coordinate as an x coordinate in a picture of f.

Example – Compositions With Graphs
Consider two graphs below of f and g.



- To evaluate $f(g(2))$, we first evaluate $g(2)$. From the graph of g, we see that $g(2) = 4$. So $f(g(2)) = f(4) = 3$.
- To evaluate $g(f(2))$, we first evaluate $f(2)$. From the graph of f, we see that $f(2) = 2$. So $g(f(2)) = g(2) = 4$.

5.2 Recognizing Compositions

As we've seen, given two functions, there are two ways we can compose them. We can also reverse the process; given a single function, we can recognize and write it as a composition of functions. Any function f is a composition of itself and the *identity* function $g(x) = x$. We've been writing functions all along as *trivial* compositions; $f(x)$ is "f of x", where x is itself a function (a boring one, but still a function!) So a composition is not always unique, that is, it is possible that a function can be written as a composition in more than one way.

Example – Recognizing Compositions

The function $f(x) = (x-2)^4 + 3$ is a composition $h(g(x))$ where $g(x) = x - 2$ and $h(x) = x^4 + 3$. We can check this:

$$h(g(x)) = h(x-2) = (x-2)^4 + 3.$$

One will require a bit of creativity and intuition when recognizing a composition. It may be helpful to think of needing one "outside function" and one "inside function". Here, the outside function is $x^4 + 3$ and the inside function is $x - 2$.

Taken together:

$$f(x) = (x - 2)^4 + 3.$$

We could have also set $h(x) = (x-2)^4$ and $g(x) = x+3$, and then $f(x) = g(h(x))$ (check this yourself). In colors,

$$f(x) = \left((x - 2)^4\right) + 3$$

Suppose $H(x) = 7\sqrt{x} - 4$. Then we find $f(x) = 7x - 4$ and $g(x) = \sqrt{x}$ so that

$$f(g(x)) = f(\sqrt{x}) = 7(\sqrt{x}) - 4 = H(x)$$

In general, when given a function and asked for a composition, we want to find an "inner" function and an "outer" function, then we can check our composition. Another way to think about it is: we find an inner function, and replace it with an x. The resulting function is the outer function. For example

$$f(x) = \sqrt{x+5} + 2.$$

We can make $x + 5$ the inner function, and replacing it with an x we get an outer function $\sqrt{x} + 2$.

5.3 Domain of Compositions

Lastly, let's look at the domain of a composition. Consider a composition $f \circ g$. The domain of $f \circ g$ is all inputs x such that x is in the domain of g and $g(x)$ is in the domain of f. That is, there are two things to check; the validity of an input x for $f \circ g$ relies on g, so that the output $g(x)$ is defined, and further, $g(x)$ must be in the domain of f so that $f(g(x))$ is defined. This all sounds a bit confusing, so let us do some examples.

Example – Domain Let $g(x) = x^2 - 1$ and $h(x) = \sqrt{x-6}$. Let's find the composition $g \circ h$ and its domain.

$$(g \circ h)(x) = g(\sqrt{x-6}) = (\sqrt{x-6})^2 - 1.$$

Notice, we may simplify further to obtain $(g \circ h)(x) = (x-6)-1 = x-7$. However, notice that by simplifying, one would incorrectly assume that the domain of $(g \circ h)$ is all real numbers. However, if we write $g \circ h$ in the non simplified form $(g \circ h)(x) = (\sqrt{x-6})^2 - 1$, we may notice that we cannot have any number x less than 6 as an input, otherwise $\sqrt{x-6}$ would be undefined. That is, the domain of h, namely $[6, \infty)$ plays an important role here. Since the range of $\sqrt{x-6}$ is all real numbers, and the domain of g is all real numbers, we do not need to worry about the output $h(x)$. So the domain of $g \circ h$ is $[6, \infty)$. In total we have

$$(g \circ h)(x) = x - 7$$
$$\text{Domain: } [6, \infty)$$

Let's do a more complicated example.

Example – Domain (Harder)

Consider the functions $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{11}{x}$. Find and simplify $(f \circ g)(x)$. Then, find the domain.

At this point, we should be able to evaluate a composition. Here it's a bit of work; the algebra can be tricky!

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{11}{x}\right) \\
&= \frac{\left(\frac{11}{x}\right)}{\left(\frac{11}{x}\right) + 1} \\
&= \frac{\left(\frac{11}{x}\right)}{\frac{11}{x} + \frac{1}{1} \cdot \frac{x}{x}} && \text{(Combine terms, LCD)} \\
&= \frac{\left(\frac{11}{x}\right)}{\frac{11+x}{x}} \\
&= \left(\frac{11}{x}\right)\left(\frac{x}{11+x}\right) && \text{(Keep, Change, Flip)} \\
&= \frac{11}{11+x}
\end{aligned}$$

Now note, we have a rational function. The domain is **not** simply obtained from finding values for which the denominator is not equal to 0. That is, the domain is **not** simply $(-\infty, -11) \cup (-11, \infty)$. This is a *part* of the domain of $(f \circ g)$. To find the domain, we go back to when we plugged $g(x)$ inside of f:

$$f(g(x)) = \frac{\left(\frac{11}{x}\right)}{\left(\frac{11}{x}\right) + 1}.$$

In this form, we find values of x which we must exclude from the domain. First, we have $\frac{11}{x}$ in the equation. This implies that $x \neq 0$, as then the function would be undefined. Notice that this issue does not appear in the simplified version of the composition! Then we notice that we have a rational function, so we must check the denominator:

$$\begin{aligned}\frac{11}{x} + 1 &\neq 0\\ \frac{11}{x} &\neq -1\\ 11 &\neq -x\\ -11 &\neq x\end{aligned}$$

So we have two excluded values, -11 and 0. Thus the domain of the composition is $(-\infty,-11)\cup(-11,0)\cup(0,\infty)$.

The above example is some work, but one needs to keep their head straight and obey algebra rules.

Example – Domain

Let $p(x)=\sqrt{x+1}$ and $q(x)=x^2+5$. Find $(q\circ p)(x)$ and find the domain.
We evaluate:

$$\begin{aligned}(q\circ p)(x) &= q(p(x))\\ &= q(\sqrt{x+1})\\ &= (\sqrt{x+1})^2+5\\ &= x+1+5\\ &= x+6.\end{aligned}$$

Again, while $(q\circ p)(x)=x+6$, a line, the domain is **not** $(-\infty,\infty)$. We focus on when we plugged in:

$$(\sqrt{x+1})^2+5.$$

Here, we have to make sure that $\sqrt{x+1}$ is defined. An even root function is defined only when the inside evaluates to a number greater or equal to 0. That is in this case, $\sqrt{\blacksquare}$ is only defined when $\blacksquare \geq 0$. So we solve $x+1\geq 0$ which yields $x\geq -1$. So in interval notation our domain is $[-1,\infty)$.

Suppose we are given a function already composed:

Example – Domain of a Complicated Function

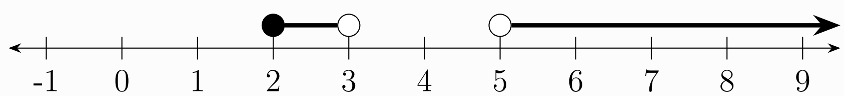
Let

$$f(x) = \sqrt{\frac{x-2}{(x-5)(x-3)}}$$

Now here, one needs to reason through. What do we know:

- The domain of any square root function $\sqrt{\blacksquare}$ contains all x values which satisfy $\blacksquare \geq 0$.
- There is a rational function inside the square root. We need to ensure that we find the excluded values; find the x values which make the denominator 0.

Thus, we need to solve $\frac{x-2}{(x-5)(x-3)} \geq 0$. Luckily, we've learned this before with rational inequalities. Setting the numerator equal to 0 we obtain a closed circle at $x = 2$, and setting the denominator equal to 0 we obtain open circles at $x = 5$ and $x = 3$. We find test values, and end up with (this work is left to the reader):



The domain in interval notation is $[2, 3) \cup (5, \infty)$.

Summary

List of things you need to know.

- Compositions are written in two ways $(f \circ g)(x) = f(g(x))$.
- Evaluate compositions of functions for certain values, or give a general formula.
- To find the domain of a composition, find excluded values when you plug in a function. Do not find the domain simply based on the simplified version of a composition.