

Rational Summer 2023

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Math 151 - 901

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5 Rational Equations, Inequalities, and Functions

In this chapter we will go over how to solve rational equations, rational inequalities, and how to understand the behavior of rational functions.

5.1 Rational Equations

Consider the following equation:

$$\frac{1}{x} = 0. \tag{1}$$

Equation 1 is a *rational* equation. Notice that the equation has **no solution**; any value other than 0 for x will lead to a nonzero number, and $x = 0$ cannot be a solution, since $\frac{1}{0}$ is undefined. In this case we say that $x = 0$ is an *excluded value*; a value that cannot be a solution to the original equation. In a rational equation, the excluded values will come from setting the denominators equal to 0.

Of course, it is not always the case that rational equations have no solutions. However, we must be weary of excluded values, that is, we may obtain a plausible solution x through the use of some algebra techniques, however x may cause some term in the

original equation to be undefined. This is very similar to equations with radicals. Let's see an example.

Example – Excluded Values Matter. Solve the following equation for x :

$$-\frac{2}{x+5} = -\frac{8}{4x+20} + 3.$$

Solution. The first step is to always clear the denominators (by using the least common denominator, or LCD, of all the fractions.) Notice that we can simplify one of the denominators on the right hand side by factoring. Then we can clean up a bit:

$$-\frac{2}{x+5} = -\frac{8}{4(x+5)} + 3$$

$$-\frac{2}{x+5} = -\frac{2}{x+5} + 3$$

Now that the denominators, namely $(x+5)$, are the same on both sides, we can multiply across by $(x+5)$ for some nice cancellation. It is at this point we note that $x+5=0 \implies x=-5$, and so -5 is an excluded value. That is, after the cancellation we can continue to solve

$$\begin{aligned}(x+5) \left(-\frac{2}{x+5} \right) &= (x+5) \left(-\frac{2}{x+5} + 3 \right) \\ -2 &= -2 + 3(x+5) \\ 0 &= 3(x+5) \\ 0 &= x+5 \\ -5 &= x.\end{aligned}$$

By our algebra, we've deduced that $x=-5$ is a possible solution. However, since $x=-5$ is an excluded value, this cannot be the case. Therefore the original equation has **no solution**. Try to plug in -5 into the original equation.

So far we've only seen rational equations with linear denominators. We can have equations with multiple solutions, and multiple excluded values. Let's look at some more difficult examples.

Example – Quadratic Denominator. Solve the following equation for x :

$$\frac{1}{x^2 - 4} - \frac{4}{x - 2} = \frac{1}{x + 2}$$

Solution. Again, the first thing we want to do is clear denominators. We can factor the quadratic denominator $x^2 - 4$ to start simplifying.

$$\frac{1}{(x - 2)(x + 2)} - \frac{4}{x - 2} = \frac{1}{x + 2}$$

To clear the denominators, we must multiply by the LCD. Notice that we can multiply across by $(x + 2)$, then by $(x - 2)$, canceling out all denominators like so:

$$(x + 2)(x - 2) \left(\frac{1}{(x - 2)(x + 2)} - \frac{4}{x - 2} \right) = (x + 2)(x - 2) \left(\frac{1}{x + 2} \right)$$

$$\frac{\cancel{(x + 2)}\cancel{(x - 2)}}{\cancel{(x - 2)}\cancel{(x + 2)}} - \frac{4\cancel{(x + 2)}\cancel{(x - 2)}}{\cancel{x - 2}} = \frac{\cancel{(x + 2)}\cancel{(x - 2)}}{\cancel{x + 2}}$$

$$1 - 4(x + 2) = x - 2$$

Note our excluded values are $x = 2$ and $x = -2$. Solving this equation gives $x = -1$, and since this value is not one of the excluded values, $x = -1$ is a solution. Plug in $x = -1$ to the original equation to check.

Example – All Together Solve the following equation for x :

$$\frac{x}{x - 7} - \frac{28}{x^2 - 49} = \frac{2}{x + 7}$$

Solution. Following all the steps highlighted earlier, we can solve for x .

$$\frac{x}{x-7} - \frac{28}{x^2-49} = \frac{2}{x+7}$$

$$\frac{x}{x-7} - \frac{28}{(x-7)(x+7)} = \frac{2}{x+7} \quad (\text{Factor})$$

$$(x-7)(x+7) \left(\frac{x}{x-7} - \frac{28}{(x-7)(x+7)} \right) = (x-7)(x+7) \left(\frac{2}{x+7} \right) \quad (\text{LCD})$$

$$\frac{x(x-7)(x+7)}{x-7} - \frac{28(x-7)(x+7)}{(x-7)(x+7)} = \frac{2(x-7)(x+7)}{x+7} \quad (\text{Distribute})$$

$$x(x+7) - 28 = 2(x-7) \quad (\text{Cancel})$$

$$x^2 + 7x - 28 = 2x - 14$$

$$x^2 + 5x - 14 = 0$$

Note the excluded values that came from factors in the denominators: $(x+7)$ and $(x-7)$ give $x = -7$ and $x = 7$ as excluded values.

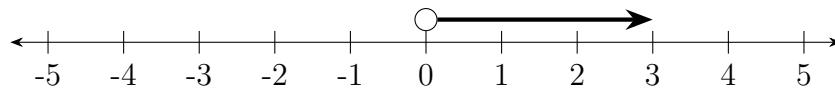
Solving the rational equation, we end up with a quadratic equation which we can solve by factoring. We get $(x+7)(x-2) = 0$, giving two solutions: $x = -7$ and $x = 2$. Since one of our solutions is an excluded value, the final solution to the original equation is $x = 2$.

5.2 Rational Inequalities

Now we focus our attention on rational inequalities. For example, there are infinitely many solutions to

$$\frac{1}{x} > 0.$$

The x values that make this inequality true are all positive real numbers, or $x > 0$. We can express this in interval notation as $(0, \infty)$, or on a number line:



Notice that we have an open circle at 0 to indicate that x is never 0.

Like rational equations, though, these can be much more complicated.

Example – All Together

Solve the following inequality for x :

$$\frac{-2}{5-x} \geq \frac{1}{x-2}.$$

Solution. First, we move everything to one side. Let's move the term on the right to the left:

$$\frac{-2}{5-x} - \frac{1}{x-2} \geq 0.$$

Now, we want to combine both fractions. We do so by finding the least common denominator. One simple way to think of this is to look at the denominator of each term and ask “what is this term missing in the denominator that is present in the other terms.” For example, the term $\frac{-2}{5-x}$ is missing a factor of $x-2$ in the denominator, and the term $-\frac{1}{x-2}$ is missing a factor of $5-x$ in the denominator. To add in the missing factors, we have to make sure we multiply both the numerator and the denominator. We have

$$\frac{-2}{5-x} \cdot \frac{(x-2)}{(x-2)} - \frac{1}{x-2} \cdot \frac{(5-x)}{(5-x)} \geq 0.$$

Note, when dealing with rational inequalities, for the most part we want to keep things in factored form. Now that the denominators are the same for each term, we can combine the terms:

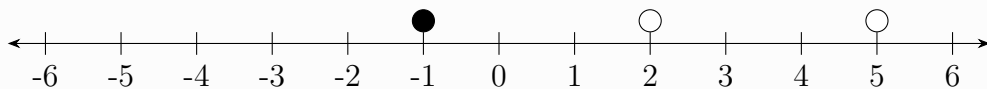
$$\frac{-2(x-2) - (5-x)}{(5-x)(x-2)} \geq 0.$$

Simplify the numerator, so that we arrive at

$$\frac{-x-1}{(5-x)(x-2)} \geq 0.$$

There is a method to the madness here. We now have the inequality in the form $\frac{\blacksquare}{\blacksquare} \geq 0$. Solving this kind of inequality, in words, is solving the question “when is this fraction positive,” while paying special attention to whether or not we are also equal to 0. Remember: a fraction is positive when the numerator and denominator are both positive or both negative. A fraction is negative when the numerator and denominator have differing sign.

To solve this inequality, we set the numerator and denominator equal to 0 to find **critical values**. Setting the numerator equal to 0 we have $-x-1=0 \implies x=-1$. Setting the denominator equal to 0 we have $(5-x)(x-2)=0$, which is 0 when $5-x=0$ or when $x-2=0$. From this we get $x=5$ and $x=2$ from the denominator. Now we set up our labeled sign chart like so:

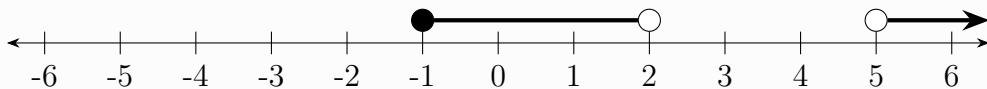


Notice, the critical values coming from the numerator have a closed circle, while the critical values coming from the denominator have an open circle. It is always the case that critical values from the denominator yield an open circle, since the denominator of a fraction can never equal 0 (undefined.) However for the critical values from the numerator, we get an open circle if the inequality sign is $<$ or $>$, and a closed circle if the inequality sign is \geq or \leq .

These critical values on the number line represent where each factor may possibly change sign. Now all that's left to do is check some test values in each interval $(-\infty, -1)$, $(-1, 2)$, $(2, 5)$, and $(5, \infty)$.

Interval	Test Value	Plug In	T/F
$(-\infty, -1)$	-2	$\frac{-(-2)-1}{(5-(-2))((-2)-2)} = \frac{1}{(7)(-4)}$	False
$(-1, 2)$	0	$\frac{-(0)-1}{(5-(0))((0)-2)} = \frac{-1}{(5)(-2)}$	True
$(2, 5)$	3	$\frac{-(3)-1}{(5-(3))((3)-2)} = \frac{-4}{(2)(1)}$	False
$(5, \infty)$	6	$\frac{-(6)-1}{(5-(6))((6)-2)} = \frac{-7}{(-1)(4)}$	True

We find test values in each interval, then plug them and check if the equality is true. After we find which intervals satisfy the inequality, we can fill in the labeled sign chart.



We are done!

So, the steps to solving a rational inequality are as follows:

1. Move everything to one side so that the inequality has 0 on one side and terms on the other.
2. Combine the terms by finding the least common denominator. Give each denominator the factor(s) it is missing from the other denominators.

3. We end up with an inequality like $\frac{\blacksquare}{\blacksquare} \geq 0$ or $\frac{\blacksquare}{\blacksquare} < 0$.
4. Once combined, set the numerator and denominator equal to 0 to find **critical values**.
 - Numerator critical values: If the inequality sign is $<$ or $>$, put an open circle. If the inequality sign is \geq or \leq , put a closed circle.
 - Denominator critical values: Always put an open circle for each critical value from the denominator.
5. There will now be several intervals on the number line. Find a test value for each interval, plug the test value into the inequality, and see if the inequality is true. **Test values cannot be critical values.**

5.3 Rational Functions

We can now talk about rational functions.

Definition (Rational Function)

A *rational function* is a function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_d x^d + b_{d-1} x^{d-1} + \cdots + b_1 x + b_0},$$

where $N(x)$ is a polynomial of degree n and $D(x)$ is a polynomial of degree d . In other words, a rational function is a ratio of polynomial functions.

In the definition the degree of the numerator or denominator is not specified; the numerator may have higher degree than the denominator, they may be equal, and so on. In Figure 1 we see some examples of rational functions and their graphs.

In this section we will examine characteristics of rational functions, including their domain and range, horizontal, vertical, and slant asymptotes, and holes. We will refer back to the functions in Figure 1 frequently as visual aids.

Definition (Asymptotes)

An *asymptote* is a line that acts as the limit for a curve; the curve can approach the line and get infinitely close, but they may never meet.

Let $f(x)$ be a rational function and a be a real number.

- A *vertical asymptote* of $f(x)$ is a vertical line $x = a$ where $f(a)$ is undefined.
- A *horizontal asymptote* of $f(x)$ is a horizontal line $y = a$ where $f(x) \rightarrow a$

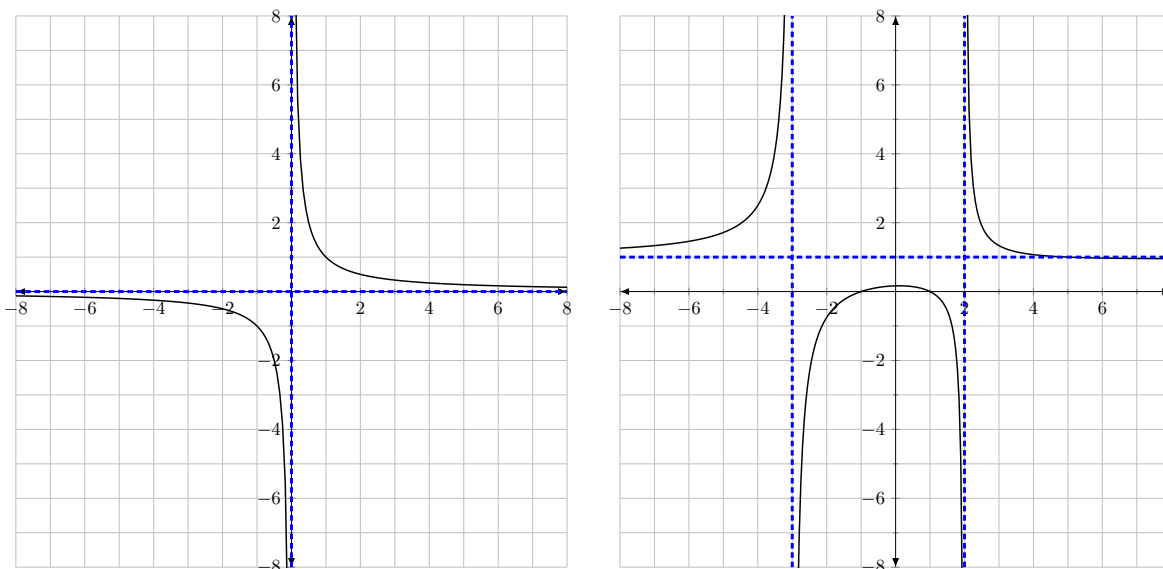


Figure 1: The function $f(x) = 1/x$ (left) and $g(x) = \frac{x^2-1}{x^2+x-6}$ (right). The *asymptotes* of the functions are dashed in blue.

as $x \rightarrow -\infty$ or $x \rightarrow \infty$. A horizontal asymptote may be crossed, but as soon as we observe far enough left or right, the function will get infinitely closer to $y = a$ but will not touch it.

- A *slant asymptote* (some times referred to as *oblique*) is an asymptote $y = mx + b$ with a slope.

There are three types of asymptotes, and we will learn how to find each type.

5.3.1 Vertical Asymptotes

Recall excluded values from rational equations. Excluded values were values of x which cause a term in an equation undefined. Specifically, they would cause ratios to be undefined by yielding a 0 in the denominator. Excluded values of rational equations represent vertical asymptotes (think of the rational terms of the equations as rational functions.) **That is to say: to find vertical asymptotes of a rational function, we set the denominator equal to 0 and solve.**

Example – Vertical Asymptotes Consider the functions in Figure 1.

For $f(x) = \frac{1}{x}$, we can quickly see that $x = 0$ is the only vertical asymptote. For $g(x) = \frac{x^2-1}{x^2+x-6}$, we must set the denominator equal to 0 and solve.

$$\begin{aligned}x^2 + x - 6 &= 0 \\(x + 3)(x - 2) &= 0\end{aligned}$$

So the equations of the vertical asymptotes for $g(x)$ are $x = -3$ and $x = 2$. You may verify this by looking at Figure 1.

Vertical asymptotes are always written in the form of an equation $x = \alpha$. On a graph, one can visualize them as dashed vertical lines.

5.3.2 Horizontal and Slant Asymptotes

Horizontal asymptotes are more involved. Let $f(x) = \frac{N(x)}{D(x)}$ where N is a polynomial of degree n and D is a polynomial of degree d . The horizontal asymptotes will depend on the relationship between the degrees of the numerator and denominator:

- If $n > d$, there is no horizontal asymptote.
- If $n < d$, there is a horizontal asymptote at $y = 0$.
- If $n = d$, then the horizontal asymptote is a ratio of the leading coefficients of the numerator and denominator, i.e. $y = \frac{a_n}{b_d}$.

Example – Horizontal Asymptotes Looking at $f(x) = \frac{1}{x}$, notice the degree of the numerator is 0 and the denominator is 1. Since $0 < 1$, there is a horizontal asymptote at $y = 0$.

For $g(x) = \frac{x^2-1}{x^2+x-6}$, the degree of the numerator and denominator are both 2; they are equal. Therefore there is a horizontal asymptote $y = \frac{1}{1} = 1$, since the L.C. of $N(x)$ is 1 and the L.C. of $D(x)$ is 1.

Horizontal asymptotes are a special case of a broader class of asymptotes, **slant asymptotes**. Horizontal asymptotes are lines with slope 0, but the slope can be nonzero. If $n = d + 1$, then the rational function has a slant asymptote. In words: if the degree of the numerator is exactly one more than the degree of the denominator, there will be a slant asymptote. To find the slant asymptote, we perform long division, and the equation is $y = \text{quotient}$.

Example – Slant Asymptote Consider the function

$$f(x) = \frac{x^2 - x + 1}{x - 1}.$$

Here $n = 2$ and $d = 1$, and since $n = d + 1$ there must be a slant asymptote. Find the equation for the slant asymptote.

Solution. To find the equation of the slant asymptote, we use polynomial long division. We obtain

$$\frac{x^2 - x + 1}{x - 1} = x + \frac{1}{x - 1}.$$

The quotient is x and the remainder is 1. The equation of the slant asymptote is $y = x$.

One doesn't have to memorize these rules if one understands what is really going on. For example, if the degree of the denominator $D(x)$ is larger than the degree of the numerator $N(x)$, then one can think of $D(x)$ as growing "faster" than $N(x)$, and as x gets really large (or really small), the gap between $N(x)$ and $D(x)$ widens. As this gap widens, the fraction $N(x)/D(x)$ shrinks, as we are dividing a numerator by a denominator which gets larger and larger. This is why if $D(x)$ has larger degree than $N(x)$, that $N(x)/D(x)$ has a horizontal asymptote at $y = 0$.

5.3.3 Holes of Rational Functions

It is important to note that there might be what we call "holes" in a rational function. These are specific points for which a function is undefined. This happens when a rational function is **defined** for us, but **not simplified**. Always remember, when you are given a rational function, make sure to factor the numerator and denominator as much as possible.

Example - Hole

Consider the rational function

$$f(x) = \frac{2(x^2 + 6x + 8)}{x^2 - 4}.$$

When we factor the numerator and denominator we get

$$f(x) = \frac{2(x + 2)(x + 4)}{(x + 2)(x - 2)}.$$

Notice that we have $(x + 2)$ in the numerator and the denominator. However, notice what happens when we plug in $x = -2$:

$$f(-2) = \frac{2((-2) + 2)((-2) + 4)}{((-2) + 2)((-2) - 2)} = \frac{2(0)(2)}{(0)(-4)}.$$

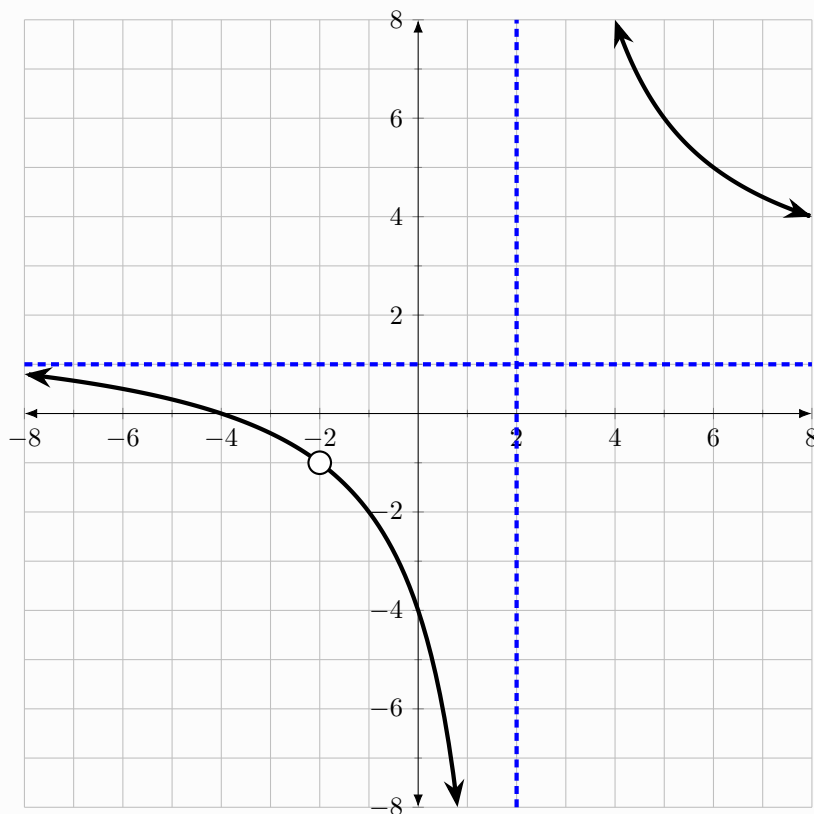
We cannot say that 0 cancels out 0, since $0/0$ does not make sense! Indeed, we can evaluate $0/1$, $0/2$, and so on, but we cannot divide 0 into 0; it is **undefined**. So while the factors $x + 2$ in the numerator and denominators of f 'cancel' when simplifying, as f is **defined**, we must make special note of this circumstance. We set the canceling factor $x + 2 = 0$, giving a hole at $x = -2$. The simplified version of f is written

$$f(x) = \frac{2(x + 4)}{x - 2}.$$

To find the y value of the hole, we plug in $x = -2$ into the simplified version of f :

$$y = \frac{2((-2) + 4)}{(-2) - 2} = -1.$$

So we say: $f(x)$ has a hole at $(-2, -1)$, and we draw an open circle on the graph.



5.3.4 Finding the Domain

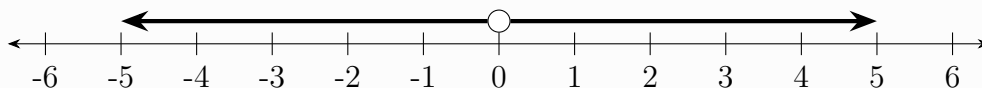
Now we have all the tools we need for finding the domain. We think of it in the following way: we start with the set of all real numbers, $(-\infty, \infty)$, and then we cut out x values for which the function is undefined. These x values are precisely the vertical asymptotes and holes.

Let's look at the functions from Figure 1.

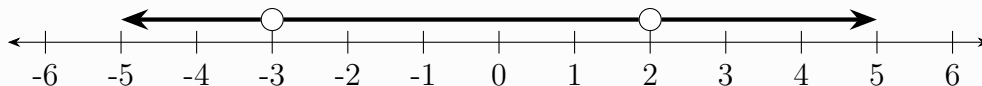
Example – Domain

Let $f(x) = 1/x$ and $g(x) = \frac{x^2-1}{x^2+x-6}$. Refer back to Figure 1 for graphs of these functions.

For $f(x)$, we know there are no holes, but we do have a vertical asymptote at $x = 0$. So $x = 0$ is not part of the domain. So we must cut it out of $(-\infty, \infty)$. Doing so yields $(-\infty, 0) \cup (0, \infty)$ in interval notation. On a number line:



For $g(x)$, we previously factored and found that $g(x) = \frac{(x-1)(x+1)}{(x-2)(x+3)}$. Setting the denominator equal to 0 we get two vertical asymptotes: $x = 2$ and $x = -3$.



In interval notation, we get $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

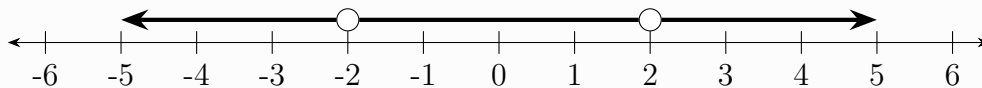
One more example that we saw earlier, this time with a hole.

Example – Domain with a hole

Let

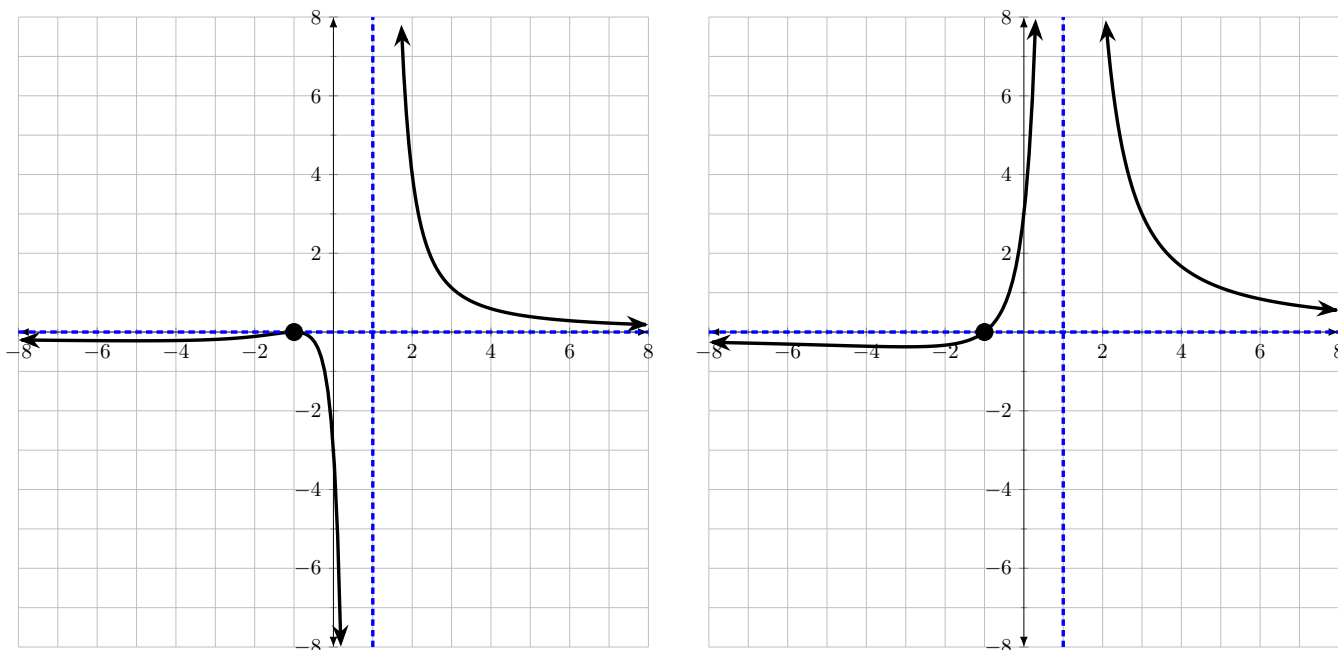
$$f(x) = \frac{2(x^2 + 6x + 8)}{x^2 - 4} = \frac{2(x+2)(x+4)}{(x+2)(x-2)} = \frac{2(x+4)}{x-2}$$

As we discussed before, instead of a vertical asymptote, we get a hole at $x = -2$, and we still have a vertical asymptote at $x = 2$. So the domain is: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. On a number line:



5.3.5 Intercept and Zero Behavior

Rational functions exhibit behaviors around the vertical intercepts and zeros. For example, let's take a look at the functions $f(x) = \frac{a(x+1)^2}{(x-1)^3}$ and $g(x) = \frac{a(x+1)}{(x-1)^2}$. (If you are curious, $a = 3$ here, but that knowledge is not useful for what we are trying to understand.)



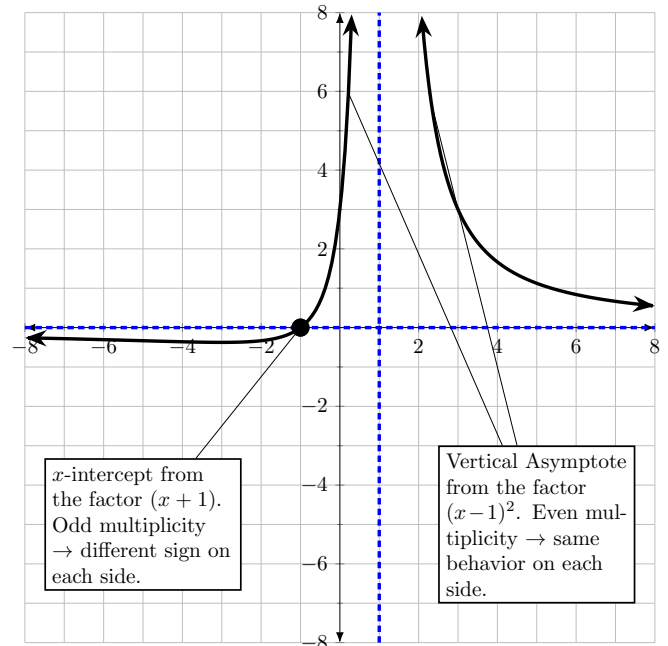
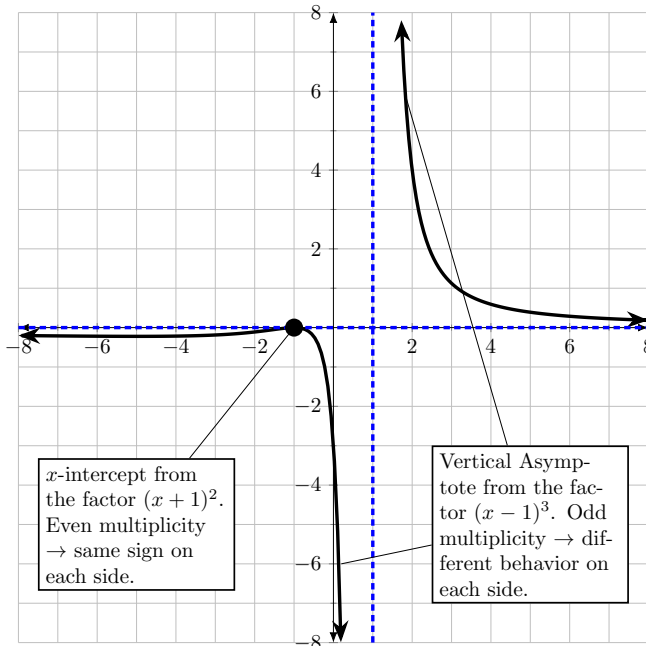
Notice that our examples both have the same x -intercept $(-1, 0)$, and the same vertical asymptote $x = 1$. However, notice the behavior of the functions. For $f(x)$ on the left of Figure ??, the sign of $f(x)$ is **the same** on the left and right of $(-1, 0)$; the function bounces at $(-1, 0)$. The behavior of $f(x)$ (going to ∞ or $-\infty$) around the vertical asymptote $x = 1$ is **different** on each side; $f(x)$ goes to $-\infty$ as it approaches from the left of $x = 1$, and goes to ∞ as it approaches from the right of $x = 1$.

Recall: after simplifying and dealing with holes, setting the numerator equal to 0 will give us x -intercepts, and setting the denominator equal to 0 will give us vertical asymptotes. Similar to polynomials, we set each factor of the numerator or denominator equal to 0. These factors have exponents, and thus we can assign *multiplicities* to not only x -intercepts (zeros, roots, same thing) but also vertical asymptotes. Here are the rules:

- If a vertical asymptote $x = \alpha$ has even multiplicity, then the function will behave the same on both sides of $x = \alpha$ (either going to ∞ on the left and right, or going to $-\infty$ on the left and right).
- If a vertical asymptote $x = \alpha$ has odd multiplicity, then the function will behave differently on both sides of $x = \alpha$ (either going to $-\infty$ from the left and ∞ from the right, or vice versa)

- If a zero $(\alpha, 0)$ has even multiplicity, then the function will have the same sign on both sides of $(\alpha, 0)$.
- If a zero $(\alpha, 0)$ has odd multiplicity, then the function will have different sign on both sides of $(\alpha, 0)$.

Looking at our examples and we can understand how the rules apply. These functions are purposefully similar, with only the multiplicities varying.



Summary

- To find the vertical asymptotes of a rational function, set the denominator equal to 0 and solve.
 - Vertical asymptotes are vertical lines indicating where the function is undefined. They are written as $x = \alpha$ where α is a zero of the denominator.
 - The behavior of a rational function around a vertical asymptote is determined by the multiplicity of the factor from which it came. Let $(x - \alpha)$ be a factor in the denominator so that $x = \alpha$ is a vertical asymptote. Recall the examples $f(x) = 1/x$ and $f(x) = 1/x^2$.
 - * If α has **odd** multiplicity, then the function behaves **differently** to the left and right of $x = \alpha$.
 - * If α has **even** multiplicity, then the function behaves **the same**

to the left and right of $x = \alpha$.

- Horizontal and Slant asymptotes. Let n be the degree of the numerator and d be the degree of the denominator.
 - if $n = d + 1$ (if the numerator has degree exactly one more than the denominator), then the function has a slant asymptote. Find the equation of the slant asymptote $y = mx + b$ by polynomial long division.
 - if $n > d$, there is no horizontal asymptote.
 - if $n = d$, there is a horizontal asymptote. Divide the leading coefficient of the numerator by the leading coefficient of the denominator to get the equation $y = c$.
 - if $n < d$ there is a horizontal asymptote at $y = 0$.
- Holes - if a factor $x - \alpha$ cancels out after simplifying $f(x)$, the graph may have a hole at $x = \alpha$.
 - If, after simplifying, $x = \alpha$ ends up being a vertical asymptote, there is no need to talk about a hole.
 - Otherwise, plug in $x = \alpha$ into the **simplified** version of f to get the y value of the hole. When graphing, there will be an open circle at (α, y) .
- Domain - Vertical Asymptotes and holes tell us what to exclude from the domain. For example, if a function has vertical asymptotes at $x = 1$ and $x = -2$, and a hole at $x = 5$, then the domain of f is, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, 5) \cup (5, \infty)$.
- x -intercepts - set the numerator equal to 0. The multiplicity of an x -intercept tells us whether the sign of the function is the same or different to the left and right of the function. Let $x = \alpha$ be an x -intercept of $f(x)$.
 - If α has **odd** multiplicity, then the sign of $f(x)$ on the left and right of α **differs** (+/- or -/+.)
 - If α has **even** multiplicity, then the sign of $f(x)$ on the left and right of α **is the same** (+/+ or -/-.)