Polynomials

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Math 151 - 901

Contents

3 Polynomials

3.1 What are polynomials?

We begin with the definition of a polynomial in the context of this class, and some of the language used when dealing with polynomials.

Definition (Polynomial)

A polynomial is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

Here the *variable* is x and the *coefficients* are the a_i. The exponent of any term in a polynomial must be a nonnegative integer. (We could say positive integer, but 0 can be an exponent.)
The *degree* of a polynomial is n, the largest exponent. The *leading coefficient*, some times abbreviated L.C., is a_n, the coefficient of the term with the largest exponent. The *constant* term is a_0. The *leading term* is a_nx^n; the leading term is the full highest degree term.

Do not let the fancy definition confuse you. Let's look at some examples.

Example – $a_2x^2 + a_1x + a_0$**.** Let's look at a familiar example: quadratic polynomials. A quadratic polynomial is a polynomial with degree 2. You may have seen the general form written as $ax^2 + bx + c$, this is fine.
Here are some examples of quadratic polynomials.

- $2x^2 + 4x + 1$. The L.C. is 2, the degree is 2, the constant term is 1 and the leading term is $2x^2$.
- $x^2 - 1$. The L.C. is 1, the degree is 2. Notice, one can say that the coefficient of x is 0, and you can instead write this polynomial as $x^2 + 0x - 1$. This line of thinking will come in handy when we discuss polynomial long division. The constant term is -1 and the leading term is x^2.

Other examples of polynomials:

- $x^3 + 2x + 1$
- $45x^{100} + x^{23} - 98x^2 - 786$
- $-x + 1$
- 5

Notice that lines are polynomial functions, and further, a number itself is a polynomial. For example, the number 5 can be written as $5x^0$, that is, any number by itself is a polynomial of degree 0.
We must be careful, however. Here are some examples of expressions which are **not** polynomials.

- $\dfrac{1}{x-1}$
- $x^2 + x^{\frac{1}{2}} + 1$
- $x^{-3} + 3x^2 - 6x + 8$
- $x^2 - x^{0.3} + 5$

A polynomial can also be written in *factored form.* We will see this a bit later when we talk about zeros, or roots. For now, let us see some examples of polynomials in factored form, and how we can get information from them.

Example – Factored Form.
Consider the polynomial

$$-2x\left(\frac{1}{2}x+1\right)(x-2)(x+3).$$

Given a polynomial in this form, we want to be able to know certain properties. Here we will focus on the **leading term.** In order to find the leading term of a polynomial in factored form, we multiply the highest term in each factor. Do not fall for the trap; there is a factor $-2x$, which you could write as $(-2x+0)$ to not confuse you. So the leading term is:

$$-2x\cdot\frac{1}{2}x\cdot x\cdot x=-x^4.$$

By finding the leading term like this, we will easily be able to find the leading coefficient and degree of a polynomial.
It is not always so trivial, however. Suppose instead we have

$$-2x\left(\frac{1}{2}x+1\right)^2(x-2)(3x+1)^3.$$

We must be careful in finding the leading term, as now some of our factors have exponents. However, this polynomial can be rewritten into something we know how to handle:

$$\begin{aligned}&-2x\left(\frac{1}{2}x+1\right)^2(x-2)(3x+1)^3\\&-2x\left(\frac{1}{2}x+1\right)\left(\frac{1}{2}x+1\right)(x-2)(3x+1)(3x+1)(3x+1)\end{aligned}$$

The leading term of the new polynomial is then

$$\begin{aligned}-2x\cdot\frac{1}{2}x\cdot\frac{1}{2}x\cdot x\cdot 3x\cdot 3x\cdot 3x&=-2x\cdot\left(\frac{1}{2}\right)^2x^2\cdot x\cdot 3^3x^3\\&=-2x\cdot\frac{1}{4}x^2\cdot x\cdot 27x^3\\&=-\frac{27}{2}x^7\end{aligned}$$

So, to get the terms we need to multiply, we take the highest degree term **in each factor and raise it to the exponent of the factor. Then, multiply all the terms.** Let's look at one final example.
Consider the polynomial

$$3x^4(2x+1)^3(x+5)^2.$$

To find the leading term, we multiply

$$\begin{aligned}3x^2\cdot(2x)^3\cdot x^2&=3x^2\cdot 8x^3\cdot x^2\\&=24x^9\end{aligned}$$

With that, we will move on to polynomial functions.

3.2 Polynomial Functions - Roots and End Behavior

By now we should be familiar with polynomial functions and how to graph them, such as lines (polynomials of degree 1) and quadratics (polynomials of degree 2). Further, in the case of quadratic functions, we should know how to factor them into linear factors. For example, $f(x) = x^2+4x+4 = (x+2)(x+2)$, and $g(x) = 3x^2+13x+10 = (3x+10)(x+1)$. The first can be factored by finding two numbers which add and multiply to 4, while the second requires the use heavier machinery (either the *ac* method, or quadratic formula, for example.)

The goal of this section is to be able to sketch a graph of a given polynomial in factored form.

Definition (Zeros or Roots)

A *zero* or *root* of a polynomial is a number α so that the polynomial evaluated at α is 0. The *multiplicity* of a root is the exponent of the factor $(x - \alpha)$ in the factored form of the polynomial.

Another seemingly terrifying definition, but something we can work through with some examples. We will see later how this comes together visually. For example, the roots or zeroes are the x-intercepts of polynomial functions.

Example (Roots and Multiplicities) Let $f(x) = x^2+4x+4$. We mentioned earlier that this polynomial factors into $(x + 2)(x + 2)$, or $(x + 2)^2$. So we can rewrite $f(x) = (x + 2)^2$. Note that $f(x) = (x + 2)^2 = 0$ if and only if $x = -2$. That is, -2 is a root of f. The multiplicity of -2 is 2, since the exponent of $(x - (-2)) = (x + 2)$ in the factored form of f is 2.
Let $g(x) = x^3 + 6x^2 + 8x$. This is a polynomial of degree 3, but it is still one we can factor.

$$\begin{aligned} g(x) &= x^3 + 6x^2 + 8x \\ &= x(x^2 + 6x + 8) \\ &= x(x + 2)(x + 4) \end{aligned}$$

So $g(x) = x(x + 2)(x + 4)$. To find the roots, we set $x(x + 2)(x + 4) = 0$, and the left hand side can only be 0 if and only if one of the three factors is 0. We can make the table:

Factor	Root	Multiplicity
x	0	1
$x+2$	-2	1
$x+4$	-4	1

Here x is a factor (can be written as $(x-0)$), and each root has multiplicity 1. Some times a polynomial comes in factored form. For example, consider the following polynomial:

$$h(x) = -2x^4(x+1)^2(x-2)(x+5)^3.$$

One may ask what the degree of this polynomial is. To find the degree, we simply add the exponent of each factor (or the multiplicities of all the roots.)

Factor	Root	Multiplicity
$(x-0)^4$	0	4
$(x+1)^2$	-1	2
$x-2$	2	1
$(x+5)^3$	-5	3

The degree of $h(x)$ is $4+2+1+3=10$.
A harder example is the following: Find all x and y intercepts of the function $f(x) = x^3+5x^2-4x-20$. This requires some creativity and experience. To find the x-intercepts, we must factor:

$$\begin{aligned} x^3+5x^2-4x-20 &= (x^3+5x^2)+(-4x-20) \\ &= x^2(x+5)-4(x+5) \\ &= (x^2-4)(x+5) \\ &= (x-2)(x+2)(x+5) \end{aligned}$$

Since the degree of this polynomial is 3, we cannot rely on our methods for factoring quadratics (ac-method, quadratic formula, etc.) Instead, we notice that we can conveniently group two portions of the polynomial, and factoring from those portions leads to a nice result. The quadratic factor x^2-4 can be factored by methods we already know. We can now find the roots by setting each factor equal to 0, giving us $x=-2,2,-5$, each with multiplicity 1. For the y-intercept, we evaluate $f(0)$ (the y-axis has $x=0$.) So $f(0)=(0)^3+5(0)^2-4(0)-20=-20$. So the y-intercept is $(0,-20)$.

The multiplicity of a root α tells us how the function behaves around the $x-axis$, at $x=\alpha$. Simply put,

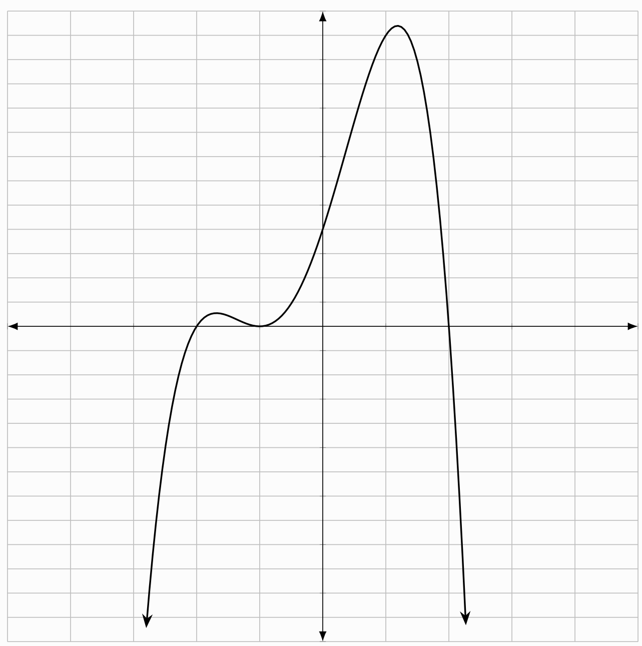
1. If a root α has even multiplicity, the function bounces on the x-axis (does not cross.)

2. If a root α has odd multiplicity, the function crosses the x-axis.

Example – Multiplicity Let $f(x) = -(x+1)^2(x-2)(x+2)$. The degree of f is 4, and the roots are

Factor	Root	Multiplicity	Behavior
$(x+1)^2$	-1	2	Bounce
$x+2$	-2	1	Cross
$x-2$	2	1	Cross

Sure enough, we can graph the function and examine the roots. We see that indeed, the function crosses the x-axis at $x = -2$, bounces at $x = -1$, then crosses again at $x = 2$.



Go back to the previous example, and plot the functions (using Desmos, for example) and see how the function behaves around the roots!

Roots of polynomials are only one piece of the puzzle. We are often interested in other characteristics, such as a polynomial's *end behavior*. How does the polynomial behave when x goes toward infinity (gets very large), or negative infinity (very small.) We will try to understand this concept graphically. Consider the two basic polynomials, $f(x) = x^2$ and $g(x) = x^3$. For $f(x) = x^2$, as $x \to -\infty$ (moves far left), f goes up to

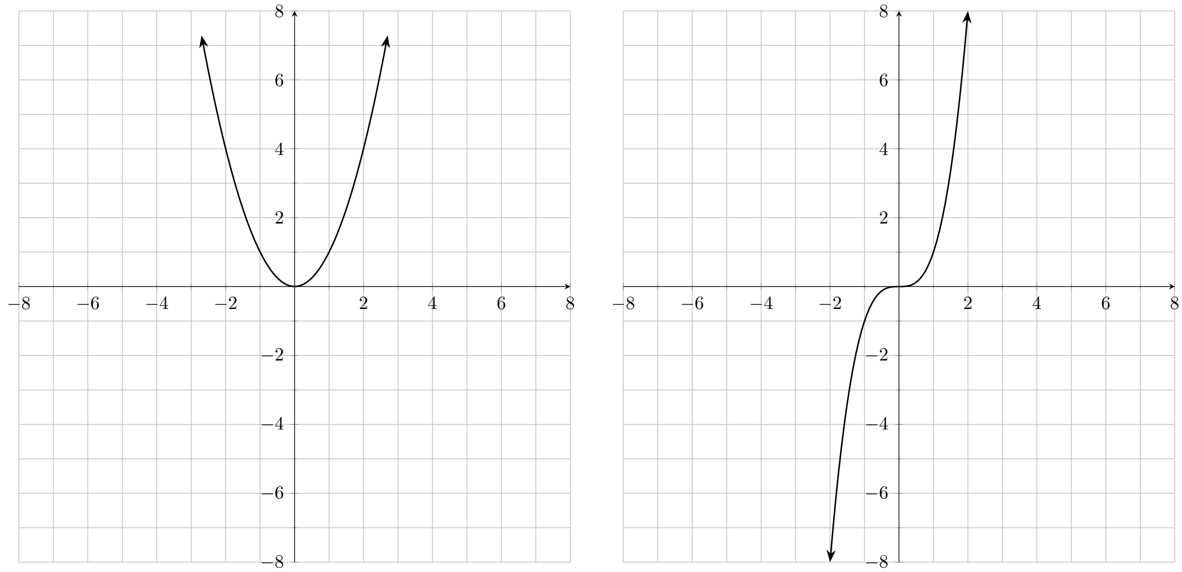


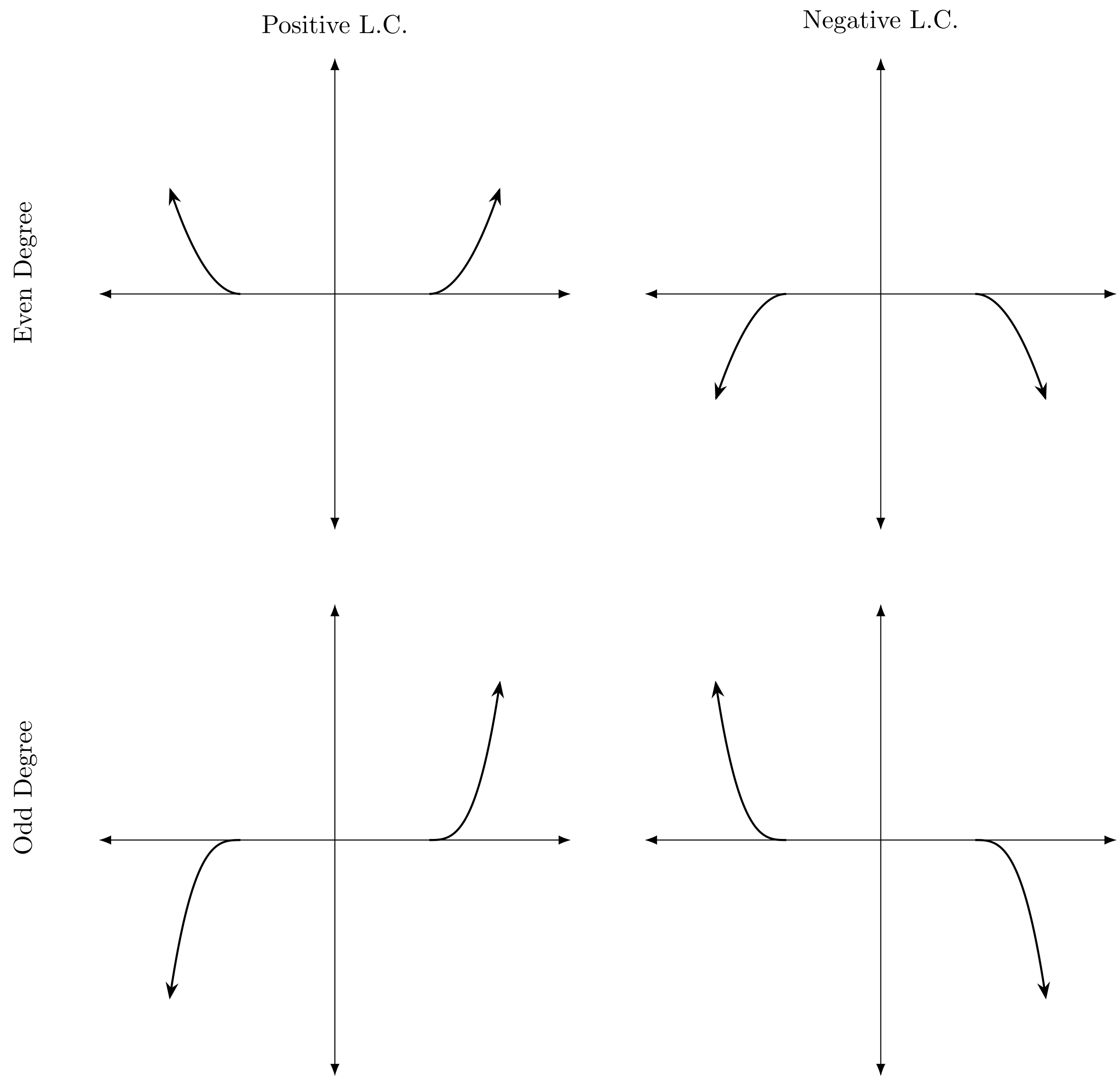
Figure 1: The functions $f(x) = x^2$ (left) and $g(x) = x^3$ (right).

infinity, and similarly when $x \to \infty$, $f(x)$ goes up to infinity as well. Here we say f **rises to the left and rises to the right.**

On the other hand, $g(x) = x^3$ displays different end behavior; as x becomes really negative, so does $g(x)$, and when x becomes really positive, so does $g(x)$. We say g **falls to the left and rises to the right.** Take a look at Figure 1.

One last thing to note is the L.C. Recall from your previous knowledge of function transformations, what happens when the L.C. is negative (reflection about the x-axis.)

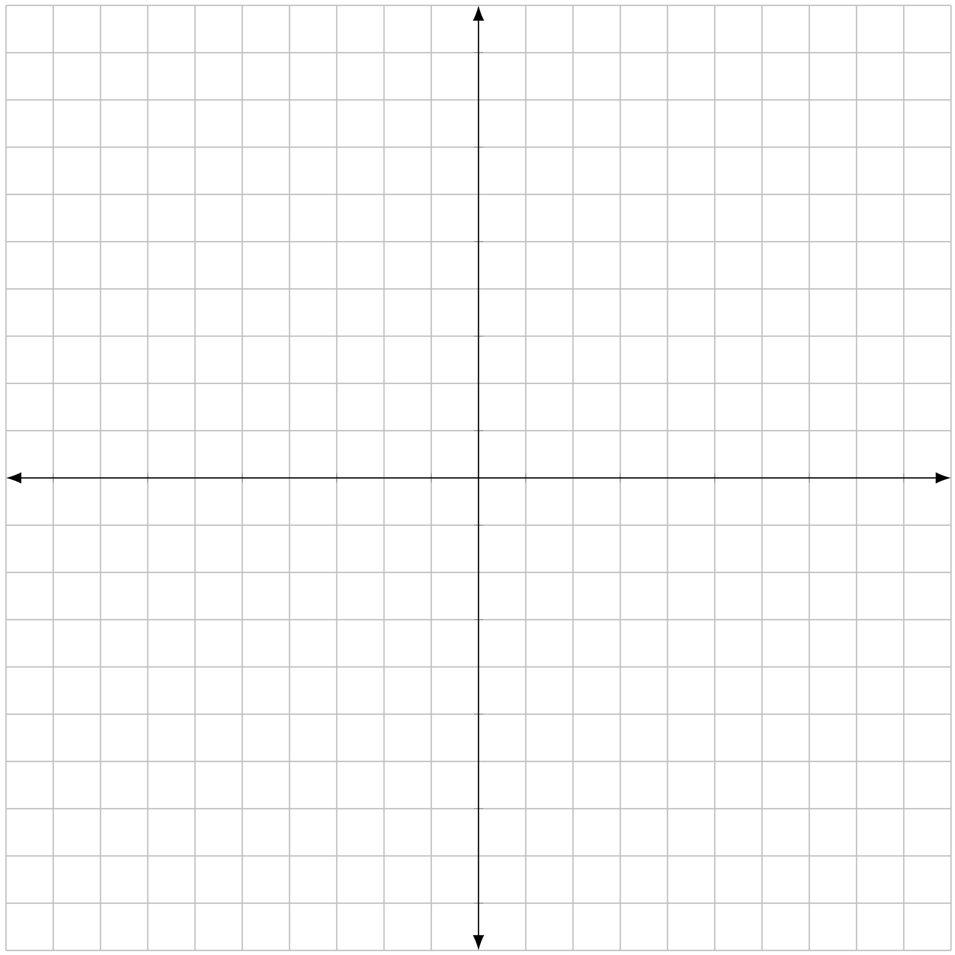
We can generalize this behavior; given any polynomial function, we will be able to tell how the output behaves as the input becomes really negative ($x \to -\infty$) or really positive ($x \to \infty$). This will depend on two things: if the degree is odd or even, and if the L.C. is positive or negative. We have the following rules:



Now try going through all of the previous examples and examine their graphs to understand root behavior and end behavior. Can you fill in the following statement for each of the previous examples?: the graph ____ to the left and ____ to the right.

3.3 Graphing a Polynomial Function

At this point, we now have all the knowledge we need to *sketch* the graph of a polynomial. Do the following example. Let $f(x) = -(x+5)^2(x-1)(x+1)^3(x-4)^2$. Sketch a graph of f. Be sure to include all x and y intercepts. **Remember: to find the y-intercept we plug in $x = 0$.**



3.4 Polynomial Long Division

So far we have been only concerned with multiplying polynomials. What happens when we divide polynomials? Or, *how* do we divide polynomials? We've divided polynomials before. For example, we know that

$$\frac{x+1}{x+1} = 1.$$

A step further, we know how to simplify something like $(x^2+6x+8) \div (x+4)$, because we can write

$$\frac{x^2+6x+8}{x+4} = \frac{(x+2)\cancel{(x+4)}}{\cancel{x+4}} = x+2.$$

Here x^2+6x+8 is the **dividend** and $x+4$ is the **divisor**.

We can learn a procedure to compute a division of polynomials.

Example – Polynomial Long Division

Divide $24x^3 + 2x^2 - 20x - 6$ by $4x + 3$.

Solution. . We first start by setting up the division. We write the dividend and divisor like so:

$$4x+3\overline{\big)\ 24x^3 \quad +2x^2 - 20x - 6}$$

The divisor is written on the outside, and the dividend is written on the inside. Now we look at the highest term in the dividend and divide it by the highest term of the divisor. Here, the highest term of $24x^3 + 2x^2 - 20x - 6$ is $24x^3$ and the highest term of $4x + 3$ is $4x$. Dividing them we obtain $6x^2$. So we write $6x^2$ on top. In order to stay neat, we can keep it in our minds that there is a column for each power of x. So, we write the $6x^2$ term above the $2x^2$ term like so.

$$\begin{array}{r} 6x^2 \qquad\qquad \\ 4x+3\overline{\big)\ 24x^3 \quad +2x^2 - 20x - 6} \end{array}$$

Now we multiply the divisor, $4x + 3$, by $6x^2$ and subtract it from the dividend. So $(4x + 3) \cdot 6x^2 = 24x^3 + 18x^2$. Notice the highest term here is $24x^3$, the same highest term as the dividend. This is not a coincidence. To subtract the result, we write $-(24x^3 + 18x^2)$ underneath the dividend.

$$\begin{array}{r} 6x^2 \qquad\qquad \\ 4x+3\overline{\big)\ 24x^3 \quad +2x^2 - 20x - 7} \\ \underline{-(24x^3 + 18x^2) \qquad\quad} \end{array}$$

Notice the $24x^3$ term cancels, and $2x^2 - 18x^2 = -16x^2$. We bring down the other terms.

$$\begin{array}{r} 6x^2 \qquad\qquad \\ 4x+3\overline{\big)\ 24x^3 \quad +2x^2 - 20x - 7} \\ \underline{-(24x^3 + 18x^2) \qquad\quad} \\ -16x^2 - 20x - 7 \end{array}$$

Now we repeat the process, with a new dividend $-16x^2 - 20x - 7$. **We keep repeating the process until the degree of the dividend is lower than the degree of the divisor.** Since the divisor $4x + 3$ has degree 1, this means we stop when the degree of the dividend reaches 0. Continuing on, we compute $\frac{-16x^2}{4x} = -4x$ and repeat.

$$\begin{array}{r|l} \multicolumn{2}{r}{6x^2 - 4x \phantom{{}-7}} \\ \cline{2-2} 4x+3 & 24x^3 \;\; +2x^2 - 20x - 7 \\ & -(24x^3 + 18x^2) \\ \cline{2-2} & \qquad\quad -16x^2 - 20x - 7 \\ & \qquad -(-16x^2 - 12x) \\ \cline{2-2} & \qquad\qquad\qquad\quad -8x - 7 \end{array}$$

Since the degree of $-8x-7$ is not less than the degree of $4x+3$, we continue.

$$\begin{array}{r|l} \multicolumn{2}{r}{6x^2 - 4x - 2} \\ \cline{2-2} 4x+3 & 24x^3 \;\; +2x^2 - 20x - 7 \\ & -(24x^3 + 18x^2) \\ \cline{2-2} & \qquad\quad -16x^2 - 20x - 7 \\ & \qquad -(-16x^2 - 12x) \\ \cline{2-2} & \qquad\qquad\qquad\quad -8x - 7 \\ & \qquad\qquad\qquad -(-8x - 6) \\ \cline{2-2} & \qquad\qquad\qquad\qquad\qquad -1 \end{array}$$

Now we stop, since the degree of -1 is 0, which is less than $4x+3$. Here, $6x^2-4x-2$ is the **quotient** and -1 is the **remainder**.
Here is an alternate way to write your work, where the negative sign is distributed at the subtraction steps and terms are added instead. If this does not resonate with you, you may disregard it.

$$\begin{array}{r|l} \multicolumn{2}{r}{6x^2 \;\; - 4x - 2} \\ \cline{2-2} 4x+3 & 24x^3 \;\; +2x^2 - 20x - 7 \\ & -24x^3 - 18x^2 \\ & \qquad\quad -16x^2 - 20x \\ & \qquad\qquad 16x^2 + 12x \\ & \qquad\qquad\qquad\quad -8x - 7 \\ & \qquad\qquad\qquad\qquad 8x + 6 \\ & \qquad\qquad\qquad\qquad\qquad -1 \end{array}$$

The answer is then written in the form

$$\text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

In this case,

$$\frac{24x^3+2x^2-20x-7}{4x+3} = 6x^2-4x-2+\frac{-1}{4x+3}.$$

The remainder is not always a polynomial of degree 0. For example, try to work out $(2x^3 + x + 1) \div (x^2 + 1)$. You should get a quotient of $2x$ and a remainder of $-x + 1$.

$$x^2 + 1 \overline{\big)\; 2x^3 \quad + x + 1}$$

So the steps to long division are:

1. Write the dividend underneath and the divisor outside.

2. Divide the highest degree term of the inside by the highest degree term of the outside.

3. Multiply the result of (2) by the divisor and subtract it from the dividend.

4. If the degree of the dividend is **less than or equal to** the degree of the divisor, then continue. Once the degree of the dividend is strictly less than the degree of the divisor, we stop and have a remainder (could be 0, could be a polynomial.)

Summary

List of things you need to know.

- A polynomial is an expression involving a variable, like x, with exponents that are non-negative integers. For example, $x^4 + x + 1$ is a polynomial.
- The degree of a polynomial is the highest exponent. For example, the degree of $x^4 + x + 1$ is 4.
- The leading coefficient (L.C.) is the coefficient of the term with the highest exponent. For example, the L.C. of $x^5 + 2x^3 + 1$ is 1.
- The leading term is the term with the highest degree. For example, the leading term of $2x^3 + x + 1$ is $2x^3$.
  - To find the leading term of a polynomial in factored form, multiply the highest degree term of each factor. For example, the leading term of $4x^2(x-1)^4(2x+3)^2$ is $4x^2 \cdot x^4 \cdot (2x)^2 = 16x^8$.
- The constant term is the coefficient of the term x^0, usually written as just a number. For example, the constant term of $x^2 + 6x + 12$ is 12.
- Roots, or zeros of a polynomial are numbers that, when plugged into a polynomial, give 0. They are the x-intercepts in the graph of a polynomial.
  - The multiplicity of a root α is the exponent of the factor $(x - \alpha)$ in the factored form of the polynomial. If the multiplicity is even, the function bounces on the x-axis at the root. If the multiplicity is odd, it crosses.
- The end behavior of a polynomial is dictated by the degree and L.C. If the degree is **even**, then
  - Positive L.C. $\nwarrow\nearrow$, the graph **rises** to the left and **rises** to the right.
  - Negative L.C. $\swarrow\searrow$, the graph **falls** to the left and **falls** to the right.

  If the degree is **odd**, then
  - Positive L.C. $\swarrow\nearrow$, the graph **falls** to the left and **rises** to the right.
  - Negative L.C. $\nwarrow\searrow$, , the graph **rises** to the left and **falls** to the right.
- The domain of any polynomial function is $(-\infty, \infty)$, or all real numbers.
- Polynomial long division algorithm.