Absolute Value and Piecewise Functions

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Math 151 - 901

Contents

2 Absolute Value and Piecewise Functions

2.1 The Absolute Value Function

Definition. Absolute Value

The *absolute value* of a number x, denoted $|x|$, is defined as follows:

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

In words, if the input x is negative, then the absolute value function multiplies x by -1 to make it positive. If the input x is positive, then the absolute value leaves it alone. For example, $|-5| = -(-5) = 5$ and $|300| = 300$.
Like quadratic functions, the minimum (or maximum) point of an absolute value function is called the *vertex*.

Graphically, the absolute value function says "for all negative values of x, I am the line $y = -x$, and for all positive values of x, I am the line x." Let's try to understand this through a picture.

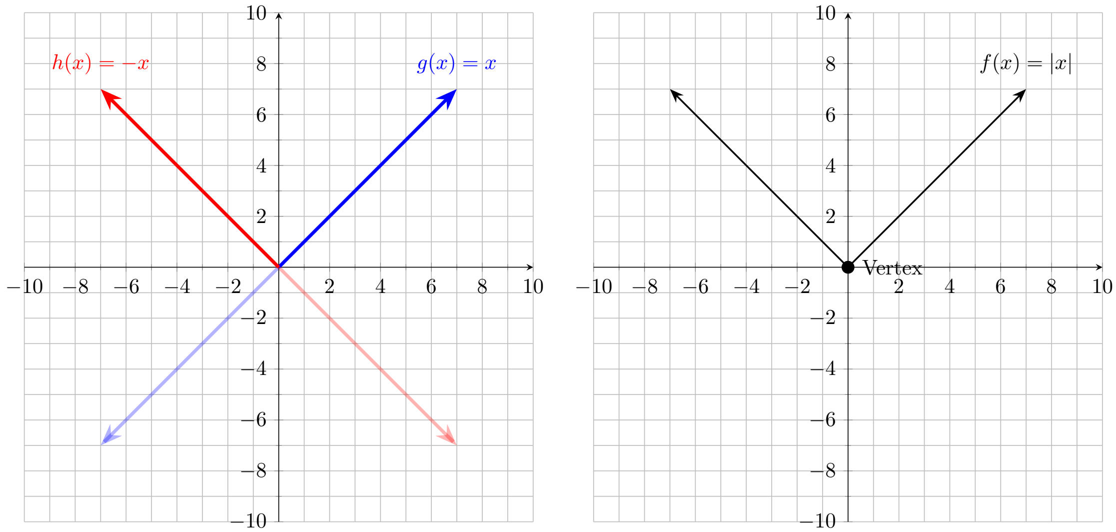


Figure 1: On the left we see the lines $g(x) = x$ and $h(x) = -x$, and on the positive x-axis we take $g(x) = x$, while on the negative x-axis we take $h(x) = -x$. On the right we have a graph of the function $f(x) = |x|$.

Absolute value functions can come in many different flavors. For example, we may have

1. $f(x) = 3|x|$
2. $f(x) = |x + 2|$
3. $f(x) = |x| + 1$
4. $f(x) = 2|x + 5| - 1$

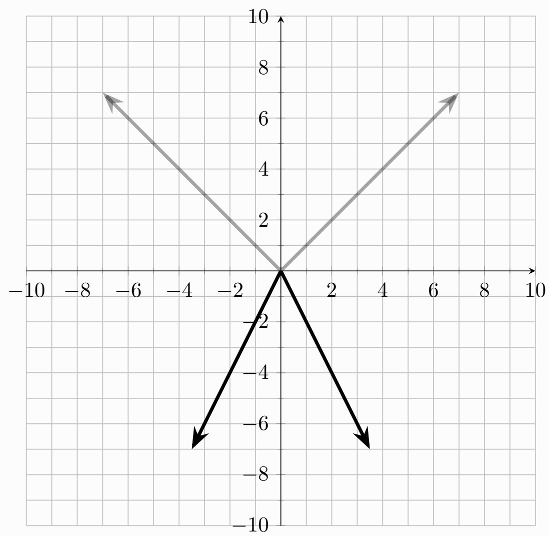
We can apply what we know about function transformations to graph an absolute value function. In the general form

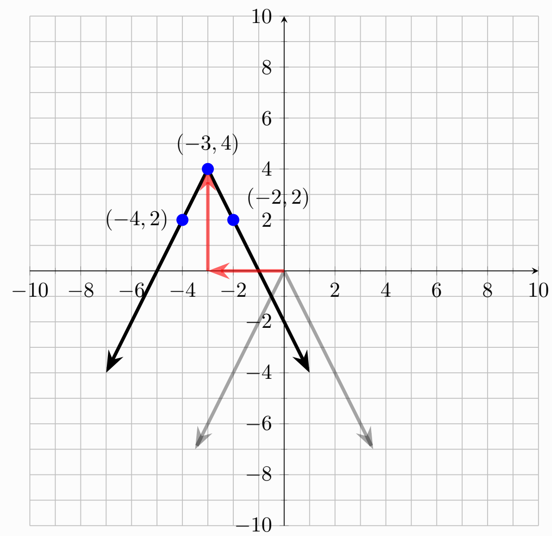
$$f(x) = a|x - h| + k,$$

we have that (h, k) is the vertex. As usual, a will vertically stretch or compress the function, and reflect it if negative. For example, $f(x) = 2|x + 5| - 1$ is $|x|$ vertically stretched by a factor of 2, shifted to the left 5 units, and shifted down 1 unit.

Given an absolute value function, one first locates the vertex (h, k). Then, find an additional point to the left or right of the vertex. Since we know the absolute value function is **symmetric**, if we have a point to the right of the vertex, we can also quickly find a point to the left. Let's do a quick example.

Example. Absolute Value Function.
Let $f(x) = -2|x+3|+4$. We will graph this function. First we vertically stretch by a factor of 2, and reflect since a is negative (-2). Then we locate the vertex; here the vertex is $(-3, 4)$. Then we can find an extra point by plugging in either -2 or -4. Doing so will yield $(-2, 2)$ and $(-4, 2)$.





Play with the following Desmos link to understand how transforming an absolute value function works: https://www.desmos.com/calculator/cu0rctptin.

2.2 Piecewise Functions

Review; functions, linear and quadratics, domains and ranges.

Definition (Piecewise Function)

A function is a *piecewise function* (some times referred to as a *piecewise defined* function) is a function defined by multiple sub functions. The sub functions are defined on segments of the domain.

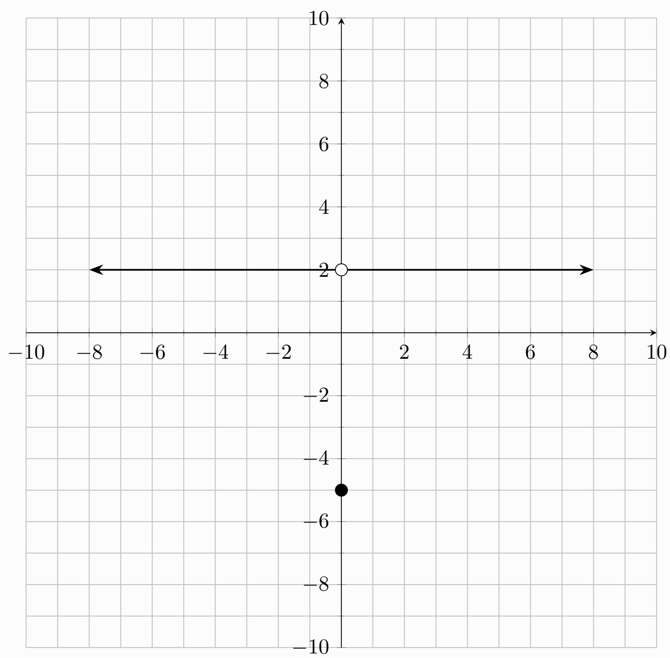
Let's understand what a piecewise function is through a basic example.

Examples. Piecewise Functions – Easy. Suppose that the function f is defined, for all real numbers, as follows.

$$f(x) = \begin{cases} 2 & \text{if } x \neq 0 \\ -5 & \text{if } x = 0 \end{cases}$$

1. Evaluate the function f for $x = -1$, $x = 0$, and $x = 1$.
2. Graph the function f.

Solution. To evaluate a piecewise function for a given input, we must first determine where that input is in the domain. We have $f(-1) = 2$, since $-1 \neq 0$. Continuing, $f(0) = -5$, since $0 = 0$. Lastly, $f(1) = 2$, since $1 \neq 0$. Now we can move on to graphing f.



Notice that the function f outputs the value 2 for any input, except for $x = 0$, for which it outputs 5. Graphically, we draw the line $y = 2$, with an empty circle at $(0, 2)$, and a filled in circle at $(0, -5)$. This indicates the behavior of the function at $x = 0$, as we described; the function does not take on the value 2 at $x = 0$, but rather it takes on the value -5 at $x = 0$.

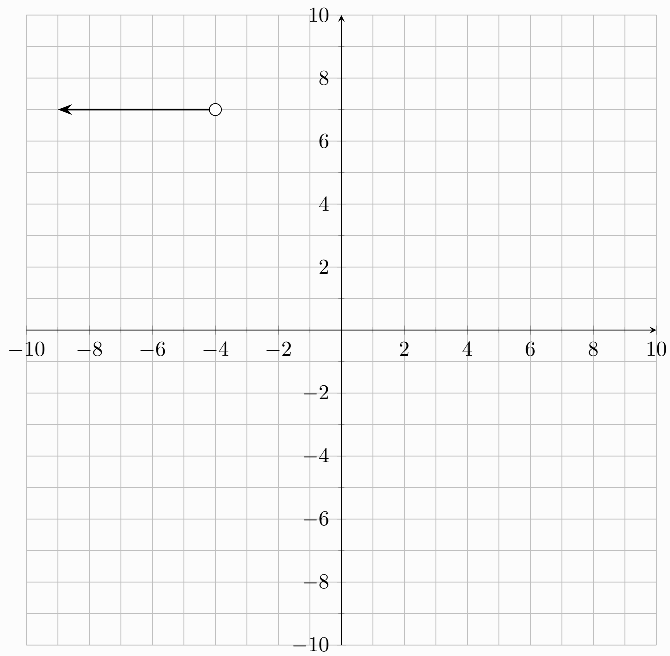
In order to understand and graph more complicated piecewise functions, the ability to evaluate them at given values is important. In your homework assignments, you may see some piecewise functions like the previous example, or ones where each piece of the function is a horizontal line. Let's do a more complicated example where there are three pieces (horizontal line, sloped line, and quadratic.)

Examples. Piecewise Functions – Hard. Suppose the function q is defined, for all real numbers, as follows.

$$q(x) = \begin{cases} 7 & \text{if } x < -4 \\ x^2 - 9 & \text{if } -4 \leq x \leq 4 \\ 2x - 1 & \text{if } x > 4 \end{cases}$$

Graph the function q. What is the domain and range of q?

Solution. Notice that our function q is three pieces, a constant function from $-\infty$ to -4, a quadratic function between -4 and 4, and a sloped line from 4 to ∞. The constant function is straightforward, we draw the horizontal line $y = 7$ from the left up until -4, where we put place open circle, since $q(x) = 7$ only when $x < 4$.

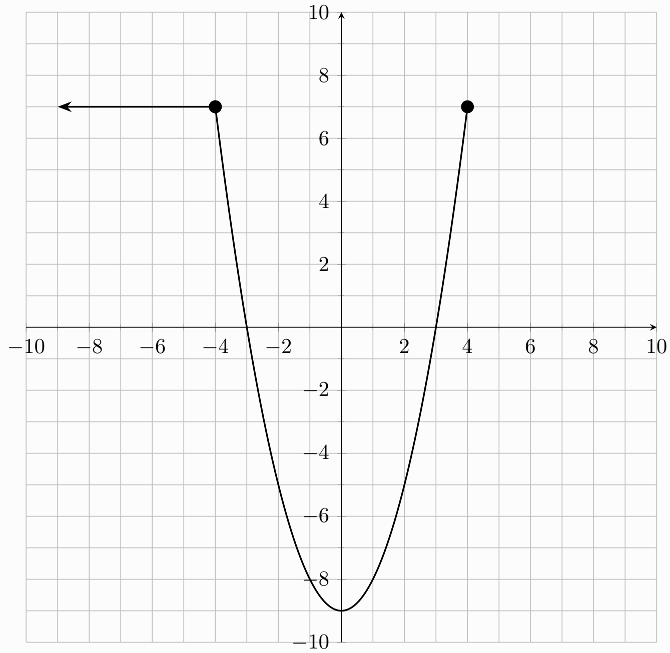


Next, we will handle the quadratic part: $y = x^2 - 9$. We know that this quadratic has vertex $(0, -9)$. We can tell this in three different ways:

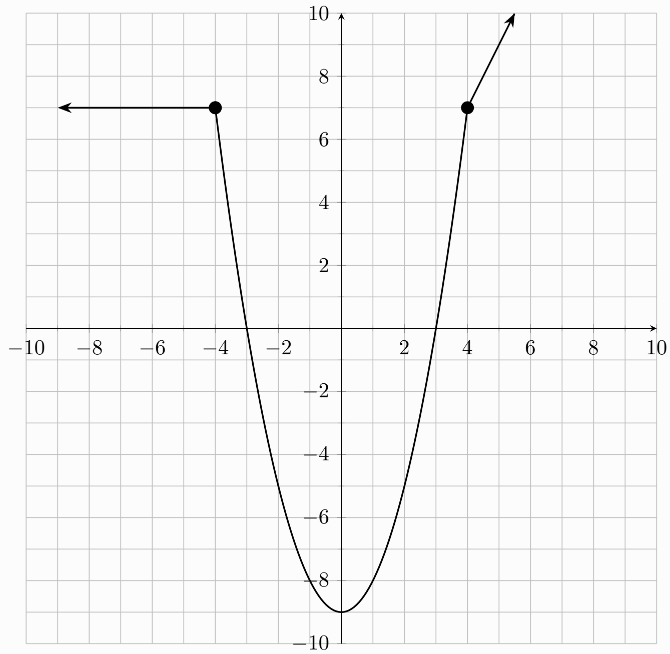
1. $y = x^2 - 9$ is x^2 shifted down by 9.
2. The x coordinate of the vertex (h, k) of any quadratic function is $h = \frac{-b}{2a}$. In this case $b = 0$ and $a = 1$. So $h = 0$, and plugging this in we get $(h, k) = (0, -9)$.

3. Recall the vertex standard form: $a(x-h)^2+k$, where the vertex is (h,k). Here we have $y = 1(x-0)^2 + (-9)$. So the vertex $(h,k) = (0,-9)$.

Now that we have the vertex, we need to evaluate the quadratic on the edge points (namely, $x=-4$ and $x=4$). Doing so gives 7 for each point: $(4)^2-9 = (-4)^2-9 = 7$. Since $q(x) = x^2-9$ for $-4 \le x \le 4$, we put filled circles on $(-4,7)$ and $(4,7)$, and using our vertex we are now able to draw in the partial parabola.



Notice that the open circle we had previously is now filled in! This is because $x^2-9=7$ at $x=-4$, that is, the horizontal piece and the quadratic piece both share a point.
To finish off, we have $q(x) = 2x-1$ for $x > 4$. We need at least two points to plot a line, one of them having $x=4$. Remember, if we want to calculate $q(4)$, we use the quadratic piece; plugging $x=4$ into $2x-1$ is only done to find where the open circle is located. Plugging in $x=4$ to $2x-1$ we obtain $2(4)-1=7$. So we would place an open circle at $(4,7)$, but since there is already a filled in circle, we leave it as is (x^2-9 accounts for $(4,7)$). Using $x=5$ for our second point, we get $2(5)-1=9$.



We have graphed the piecewise function in its entirety. Now what about the **domain and range**?

Notice that we can plug in any real number into q. Conceptually, in interval notation, we can see that for $(-\infty, -4)$, $f(x) = 7$ and for $(4, \infty)$, $q(x) = 2x - 1$. The quadratic closes the gap by taking on the values $[-4, 4]$. So, in total the domain is $(-\infty, -4) \cup [-4, 4] \cup (4, \infty) = (-\infty, \infty)$. We can figure out the range quickly. The quadratic portion has a minimum value of -9, and increases until 7. The sloped line portion starts at $(4, 7)$, and increases to infinity. So we have the range of q as $[-9, \infty)$.

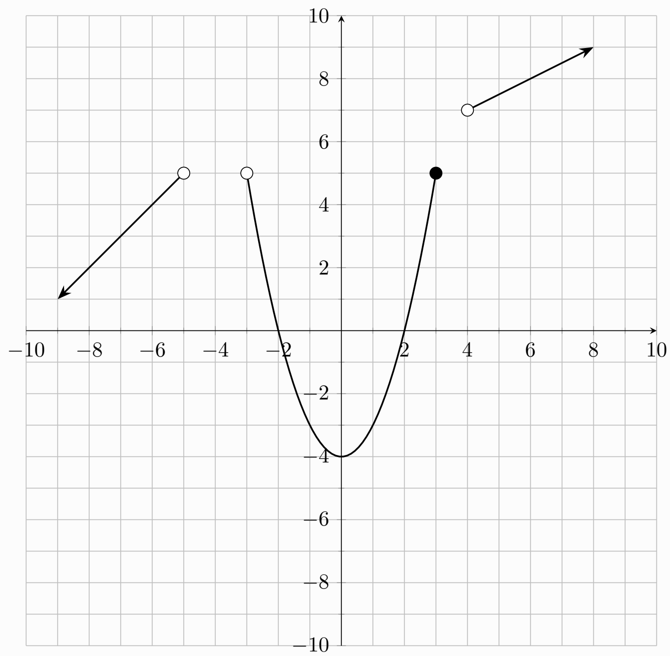
Notice that the function we obtained in the previous example is *continuous*. That is, you can draw or trace the function, without lifting off the paper. There are no gaps or jumps in the graph.

We will finish off with a modification of the previous example, that will have a more complicated domain and range.

Examples. Piecewise Functions – Hard. Suppose the function f is defined, for all real numbers, as follows.

$$f(x) = \begin{cases} x + 10 & \text{if } x < -5 \\ x^2 - 4 & \text{if } -3 < x \leq 3 \\ \frac{1}{2}x + 5 & \text{if } x > 4 \end{cases}$$

Graph the function f. What is the domain and range of f?
Solution. Immediately, one should notice that the domain will be broken into pieces: we have three pieces, each living on $(-\infty, -5)$, $(-3, 3]$, and $(4, \infty)$. Using what we now know, we should be able to graph this function. (Evaluating piecewise functions, picking the proper points to evaluate, and placing open or filled circles, calculating the number of points needed to plot a piece, etc.).



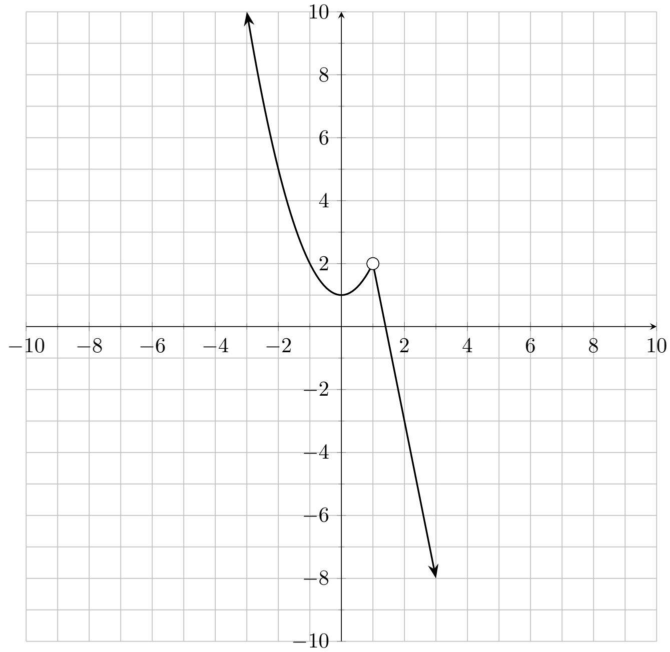
Looking at the y-axis, there are also gaps in the range. Can you see why the domain of f is $(-\infty, -5) \cup (-3, 3] \cup (4, \infty)$ and the range is $(-\infty, 5] \cup (7, \infty)$?

The function in the previous example is **not** continuous. There are two gaps, one between each piece of the piecewise function from left to right.

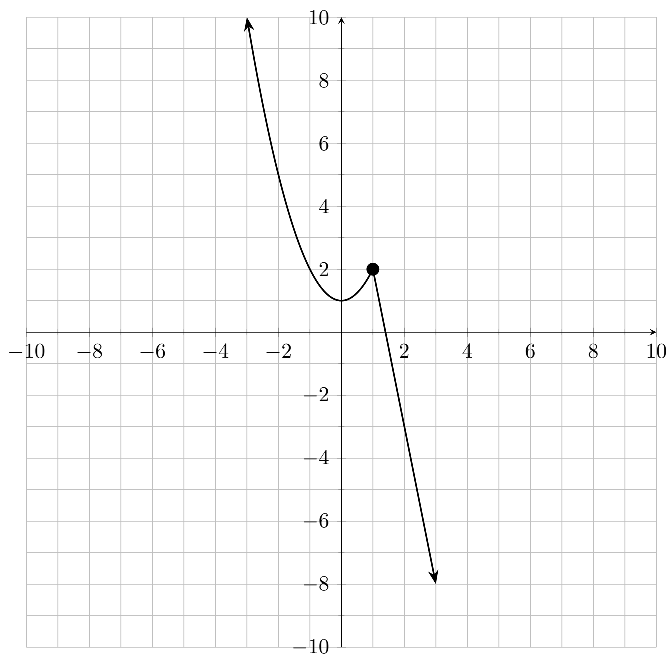
Note that you may encounter some differences in your homework problems. Some times plotting a quadratic piece is not as symmetric as these examples. For example, you may need to graph the function

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 7 - 5x & \text{if } x > 1 \end{cases}$$

which looks like



This function is not continuous, however we can fill in the hole by adding a “piece” to f. That is, we can define $f(1) = 2$, and $f(x) = 2$ only when $x = 1$. This would produce which looks like



Summary

List of things you need to know.

- What is a function? How to work with functions (their names, inputs, outputs, etc.) What is the vertical line test?
- What is a piecewise function?
  - How to evaluate piecewise functions.
  - How to graph piecewise functions. (This involves graphing lines and parabolas.) Finding the proper points to plot, and where to put open and filled circles.
  - What is the domain and range of a given piecewise function.
  - Is a given piecewise function continuous – A function is continuous if you can trace the function without lifting off the paper (there are no holes, jumps, or gaps.)