Trigonometry
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Math 151 - 901

Contents

10 Trigonometry

In this chapter we will learn trigonometry, the study of angles of triangles and relations between their sides. Note that the first few pages, mainly section 11.2 on angles are **foundational.** Without a good understanding of angles, coterminal angles, reference angles, and the four quadrants, later topics will be exceedingly difficult.

10.1 Introduction

At this point in one's life, one has an understanding of what an angle is. An *angle* can be thought of as a measurement of the space between two intersecting lines, or an angle is an object formed by taking two rays (sides) which end at at common point. This measurement is commonly measured in *degrees.* As a review, we have several types of angles, depicted in Figure 1. We usually denote an angle measurement by the greek letter θ (read aloud as 'theta').

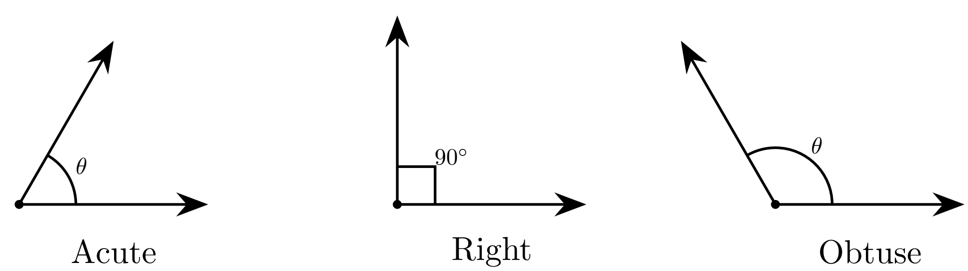


Figure 1: Different types of angles. In this picture from left to right, the first angle is 60°, the second is 90°, and the third is 120°. Acute angles are angles that measure between 0° and 90°, right angles are exactly 90°, and obtuse angles measure between 90° and 180°. There are more types of angles, such as straight angles (exactly 180°) and reflex angles ($> 180°$). The reader is encouraged to draw pictures of straight and reflex angles on their own.

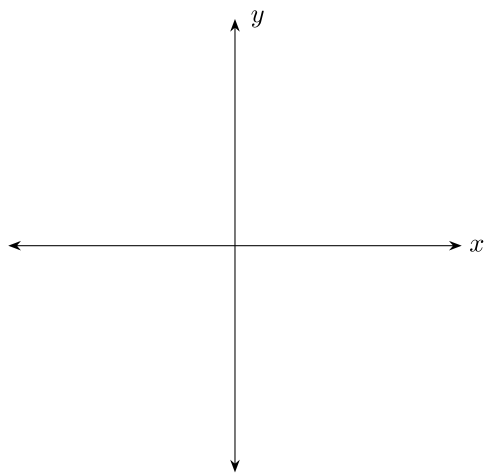
The reader is encouraged to review some basics on angles and recall knowledge of triangles from previous courses.

10.2 Angles in the xy-Plane

Instead of arbitrarily drawing lines in any orientation, we will focus on angles in the xy-plane and follow some conventions so that we can explore trigonometry in a standardized way. We will see how to sketch angles in **standard position.**

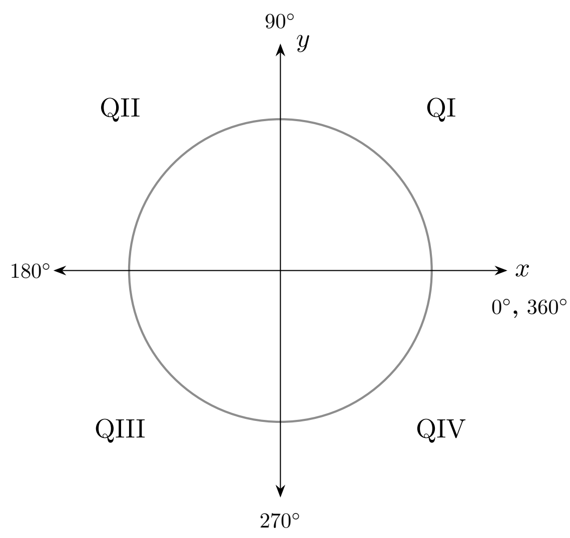
10.2.1 Sketching an Angle in Standard Position

Below we have our xy-Plane.



Like previously with functions and graphs, we have an x-axis with a positive side (right) and negative side (left). We have a y-axis with a negative side (below) and a positive side (above). Notice that the plane is divided into four sections. These sections are referred to as *quadrants.* The quadrants are labeled in order, counter clockwise starting from the positive x-axis. Later on it will be important for us to be able to tell with quadrant we are in based on the value of a given angle, or the signs of coordinates. For example, if x is positive and y is negative, then the point (x, y) is in quadrant IV (often abbreviated QIV).

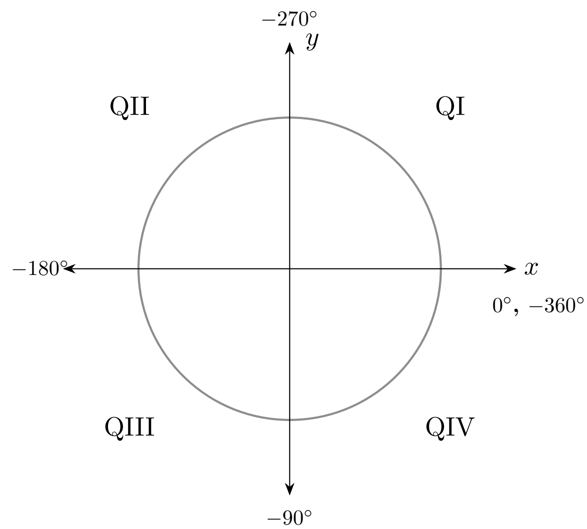
Notice that the x axis is perpendicular to the y axis; that is there are four **right angles** where the x and y axis intersect. Loosely, one can think of each quadrant as being worth 90°. Let's imagine that there is a full circle centered at the origin, so that we can better envision the rotations.



One sketches a given angle θ in the xy-plane in the following way. We first begin on the positive x-axis. To determine which quadrant we are in we see where our angle lands. Unless the angle is one of $0°, 90°, 180°, 270°, 360°$, or a multiple of those, we will lie inside a quadrant. The angle θ is in:

- QI if $0^\circ < \theta < 90^\circ$
- QII if $90^\circ < \theta < 180^\circ$
- QIII if $180^\circ < \theta < 270^\circ$
- QIV if $270^\circ < \theta < 360^\circ$.

Notice that θ can be negative, and these rules then would not apply. However, the quadrants always remain the same (their positions do not change.) If θ is positive, then we begin rotating counter clockwise (up from the x-axis). If θ is negative, we begin rotating clockwise (down from the x-axis). So we may redraw the above picture in the following perspective when dealing with negative angles.



We then draw an arc with an arrow indicating the direction of rotation, and we label the arc with θ, whatever it may be. For example, see Figure 2. Some angles may be greater than 360°, or less than -360°. That is, we are not limited to a number of full rotations; an angle can be very large or very negative, and have several full rotations before terminating. For example, the angles 400° and -510°.

To determine what quadrant a larger angle (one that is larger than 360°) is in, it is helpful to first "subtract away" the full rotations. For example, since 400° is larger than 360°, we know that there is one full rotation that lands us back on the positive x-axis. What remains is 40° more of rotation. Similarly for -510°, we rotate clockwise -360°, leaving us with a remaining -150° starting at the positive x-axis. The concept of coterminal angles will be helpful for this, but first we will mention radians.

For some terminology, the positive x-axis is called the *initial side* and the line we draw after rotation is the *terminal side.*

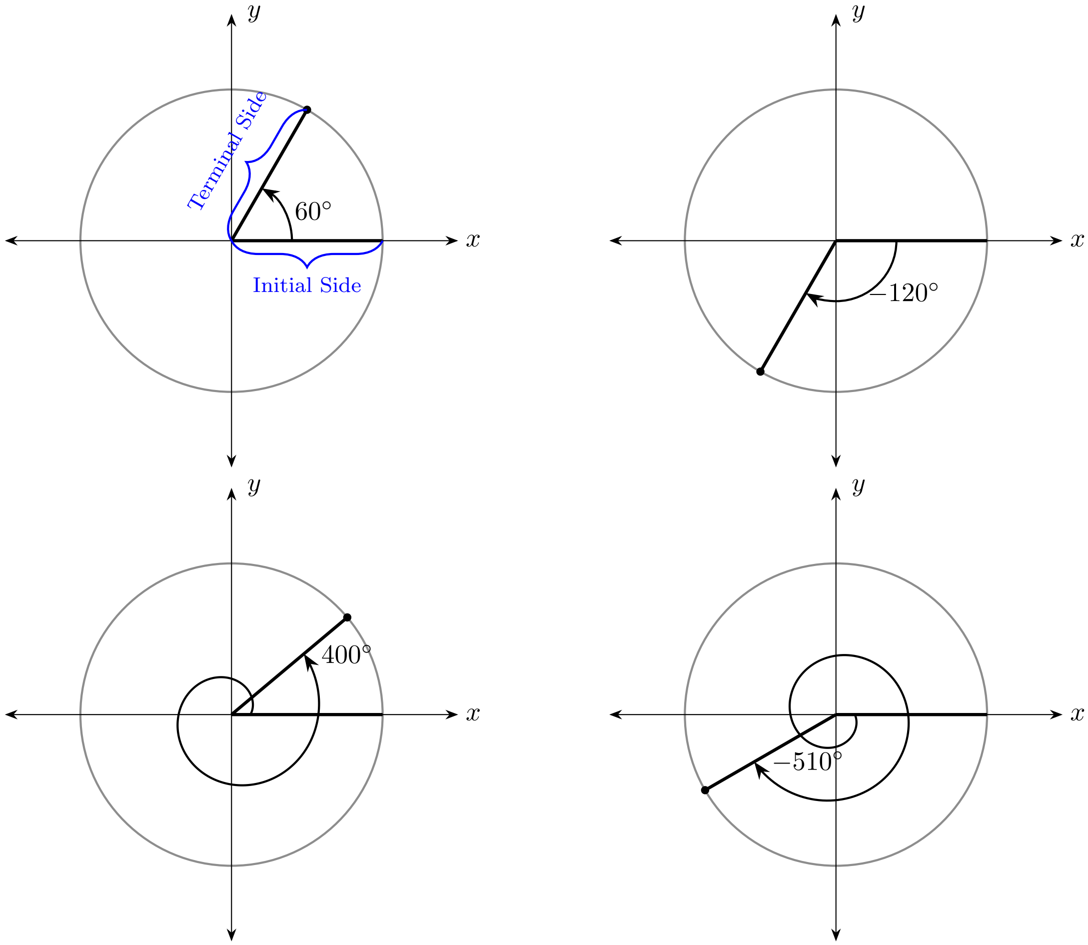


Figure 2: Four angles sketched in standard position. Start at the positive x axis, determine what quadrant the angle is in, and then rotate in the proper direction depending on whether θ is positive or negative.

10.2.2 Radians

Although understanding circles and angles in the context of a full rotation as 360° is useful; the number 360 has many factors:

$$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 72, 90, 120, 180, 360\}.$$

Several of those factors are "special" angles that we will care about later, but even numbers like 90 and 180 are factors, and as angles occur naturally. This allows us to cut up a circle into a whole number of pieces, which each piece represented by a whole number angle, and this can be done for each factor.

There is another intuitive way to understand angles, and it stems from the ratio between the radius and circumference of a circle. Recall that the radius r of a circle is the distance between the center point of the circle and the outer boundary. The circumference C of a circle is the distance around the circle; if one makes a circle with a piece of rope, then cuts the rope and lays it against a ruler, the length of the rope is the circumference. We then use the greek letter pi (π) to denote the following relationship:

$$\pi = \frac{C}{2r}.$$

This relationship is a nontrivial discovery; while throughout early human history there have been many estimations and hints at π, this relationship is a more formal result.

Solving the equation gives $C = 2\pi r$. Throughout this chapter we will be using the **unit circle**, or the circle of radius 1 centered at the point $(0, 0)$. The unit circle has circumference $C = 2\pi$. Then we can think of angles in the same way we think of 360°, except in terms of 2π. For example, half a circle in degrees is half of 360°. In radians, then, 180° is π radians, or half of 2π. Similarly, 90° is one fourth, or a quarter of a full circle. A quarter of 2π is $\frac{2\pi}{4} = \frac{\pi}{2}$, and so 90° and $\frac{\pi}{2}$ are equivalent. Understanding how angles are "portions" of a full circle, we see that viewing the circle in degrees or in radians is the same.

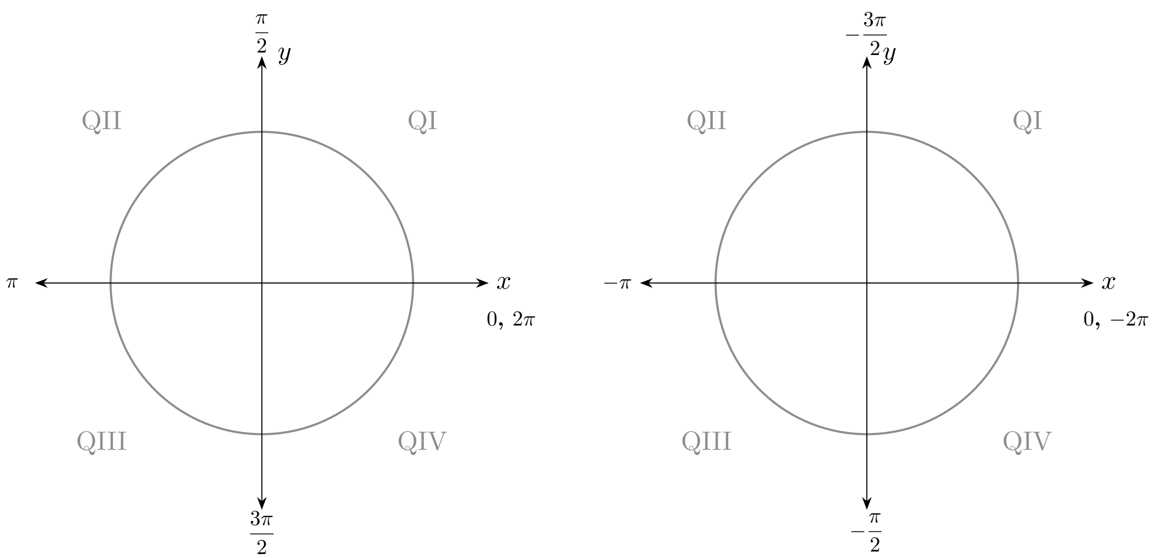


Figure 3: The plane labeled with radians instead of degrees.

10.2.3 Coterminal Angles

Definition – Coterminal

Two angles are coterminal if their terminal sides are the same when sketched in standard position.

For those already familiar with the unit circle, two angles are coterminal if they correspond to the same point (x, y) on the unit circle.

For example, consider $\theta = 150°$, which terminates in the second quadrant QII. In order to sketch $150°$, we rotate counter-clockwise until we land in QII, and then draw the terminal side. However, notice that we could have also rotated in the opposite direction; we could have rotated clockwise to QII. How many degrees would we have rotated clockwise? IF a full circle is $360°$, and we rotate counter-clockwise by $150°$, then the remaining portion of the circle is $360° - 150° = 210°$. That is, to arrive at the same terminal side as $150°$ by rotating clockwise, we rotate counter-clockwise by $210°$, or rotate by $-210°$. One could also think of this situation in the following way.

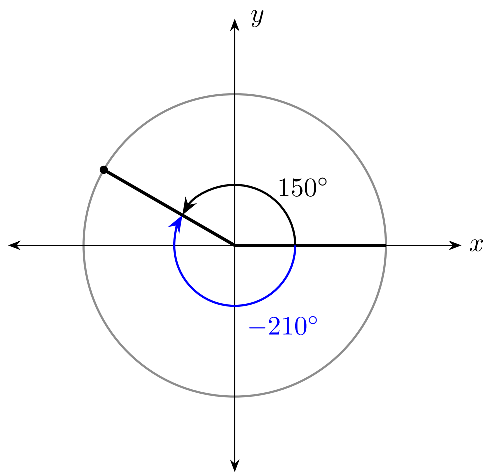


Figure 4: Two angles which are coterminal.

We know a full revolution is $360°$. Then, one can consider the terminal side of $150°$ as an initial side. That is, if we start from the terminal side of $150°$ and rotate clockwise or counter-clockwise by full revolutions, we can find many angles which are coterminal with $150°$. By starting at the terminal side of $150°$, if we rotate clockwise, then we subtract $360°$ from $150°$, and if we rotate counter-clockwise, we add $360°$.

So, from this we can see that $-210°$ and $510°$ are coterminal with $150°$. So given any angle θ, we can obtain an angle θ_{c} coterminal to θ by adding or subtracting $360°$. In radians, we replace $360°$ with 2π as before. In math:

$$\theta_c = \theta + 360° \quad \text{or} \quad \theta_c = \theta - 360°$$
$$\theta_c = \theta + 2\pi \quad \text{or} \quad \theta_c = \theta - 2\pi$$

Using these equations, we see that if we take the difference of any angle with an angle which is coterminal to it, in other words $\theta_c - \theta$, then the result must be either $360°$ or $-360°$ (2π or -2π in radians.) Then, **given any two angles, we can check if they are coterminal by taking their difference. If the difference is a *multiple* of a full revolution, then they are coterminal.**

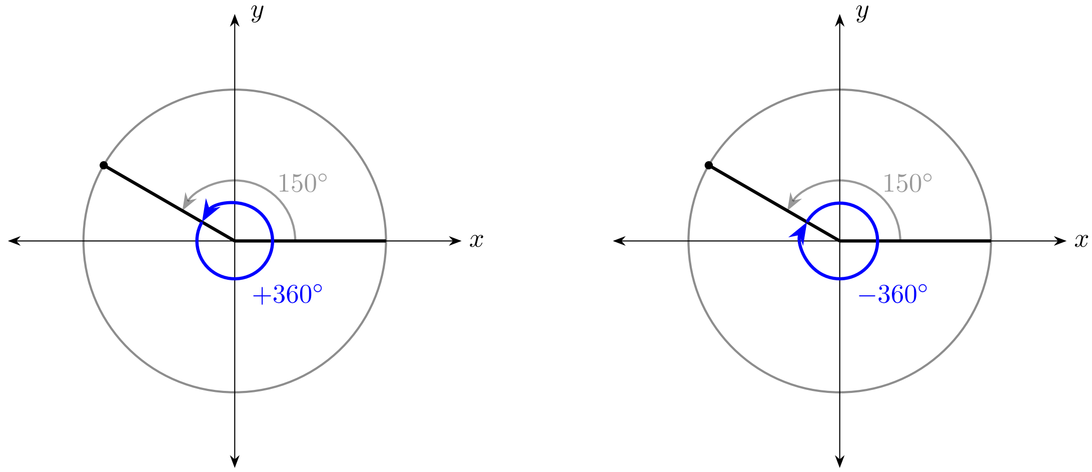


Figure 5: We start from 150°, then rotate a full revolation either clockwise or counter-clockwise. If counter-clockwise, we add 360° to the already existing 150°. If clockwise, we subtract 360°.

Example – Coterminal Angles

1. Find one positive and one negative angle which is coterminal to $\dfrac{7\pi}{3}$.

   Here, notice that $\frac{7\pi}{3}$ is larger than 2π. That is, it is more than one full revolution. To find a positive coterminal angle then, we can subtract 2π:

   $$\frac{7\pi}{3} - 2\pi = \frac{7\pi}{3} - \frac{6\pi}{3} = \frac{\pi}{3}.$$

   Of course, we could have also added 2π as a safer option. Now to find a negative coterminal angle, we will need to subtract once more.

   $$\frac{\pi}{3} - 2\pi = \frac{\pi}{3} - \frac{6\pi}{3} = \frac{-5\pi}{3}.$$

   The main take-away here is that it may take subtracting or adding full revolutions more than once to obtain a desired coterminal angle (in this case, one negative and one positive).

2. Let $\theta = 315°$. Check if the following angles are coterminal with θ.

   (a) $-45°$

   (b) $\frac{9\pi}{4}$

   (c) $\frac{23\pi}{4}$

   (d) $-405°$

Notice that θ is given to us in degrees, but we are asked to compare to other measurements in radians. We can convert θ to radians, or we can convert the other given angles to degrees. One should do what they are most comfortable with, though it may be more convenient to be comfortable with radians so that we only need to convert θ. In radians. $315°$ is $\frac{7\pi}{4}$ (check this.)

(a) $-45°$. Taking the difference, $315° - (-45°) = 315° + 45° = 360°$. ✓

(b) $\frac{9\pi}{4}$. Taking the difference with our converted value of θ, we have $\frac{7\pi}{4} - \frac{9\pi}{4} = -\frac{2\pi}{4} = -\frac{\pi}{2}$. This is not a multiple of 2π. ✗

(c) $\frac{23\pi}{4}$. Taking the difference, $\frac{7\pi}{4} - \frac{23\pi}{4} = -\frac{16\pi}{4} = -4\pi$. This is a multiple of 2π. ✓

(d) $-405°$. Taking the difference, $315° - (-405°) = 720°$. This is a multiple of $360°$. ✓

10.2.4 Reference Angles

The final topic in this introduction on angles and standard position is reference angles.

Definition – Reference Angle

The reference angle of an angle θ is the **acute** angle between the terminal side of θ and the x-axis. The reference angle is always **positive**.

Four examples are highlighted in Figure 6. Given an angle between 0 and $360°$, one can find the reference angle by understanding what information one has access to. For example, if θ is in the first quadrant, then θ is it's own reference angle. So the reference angle of $60°$ is $60°$ (or $\pi/3$.) If θ is in the second quadrant, then we must think a bit more. For example, consider $150°$. We know that we rotate counter-clockwise $150°$, and we know that half of a circle is $180°$ of a rotation. Then $180° - 150° = 30°$ is our reference angle. That is, the acute space between the terminal side of $150°$ and the x-axis is half a circle, minus what was already traveled. Another way to think about it is, we first rotate $150°$, then to travel from the terminal side of $150°$ to the x-axis, we must travel $30°$ more to reach $180°$. With similar logic we can deduce that if θ is in QIII then the reference angle is $\theta - 180°$, since once again we want to measure the space between the x-axis and the terminal side.

We can make the following table. The reader is encouraged to work out the justification for QIV.

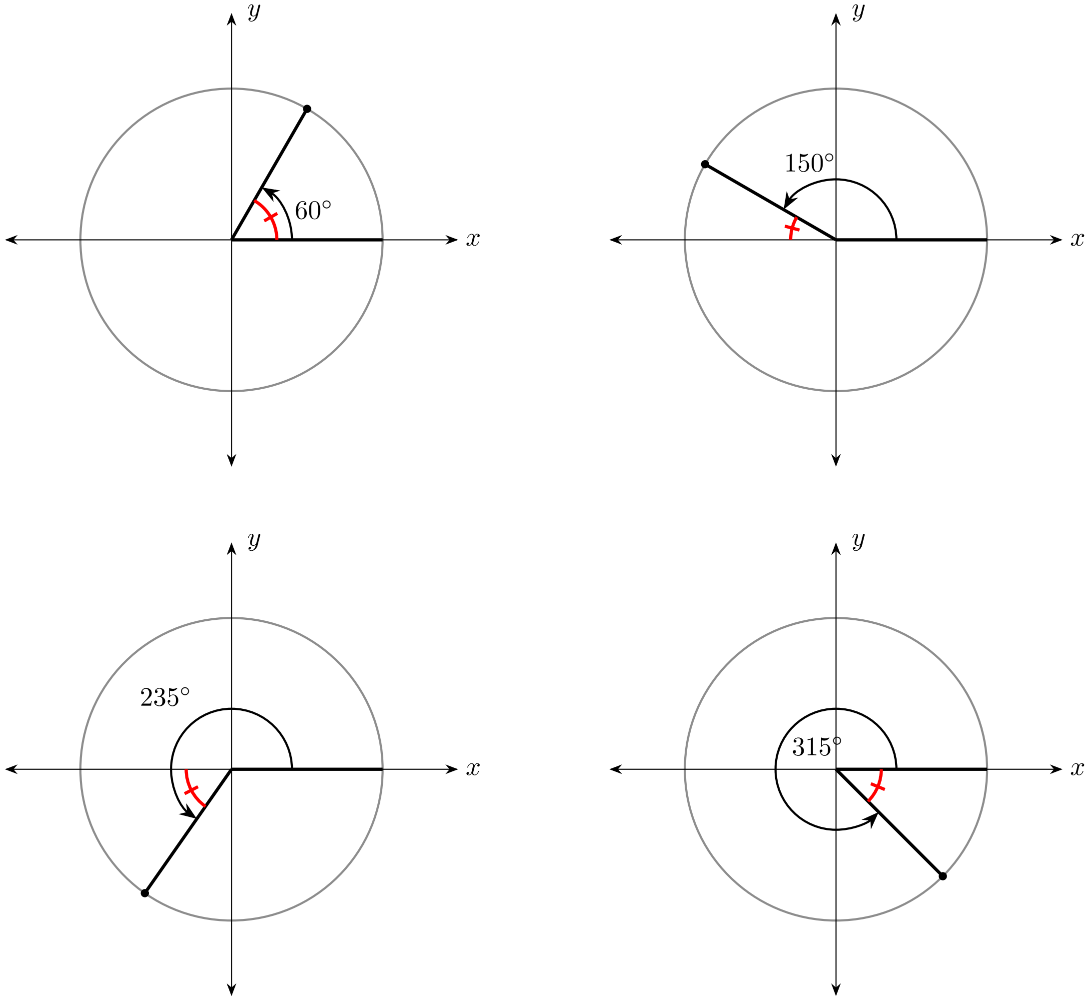
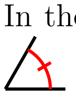


Figure 6: One angle is sketched to illustrate how reference angles work in each quadrant. In the picture, each reference angle is marked with a red arc with a dash through it,  .

θ Location	Ref Angle (DEG)	Ref Angle (RAD)
QI	θ	θ
QII	$180° - \theta$	$\pi - \theta$
QIII	$\theta - 180°$	$\theta - \pi$
QIV	$360° - \theta$	$2\pi - \theta$

Note, these formulas only work for angles which are between 0 and 360°, or 0 and 2π. If one is given an angle which does not fall in those ranges, then one must find the positive **coterminal** angle in the range. Coterminal angles share the same reference angle, as they have the same terminal side!

Example – Reference Angles

1. Here are some angles and their reference angles.

   (a) $150°$ has a reference angle of $180° - 150° = 30°$.

   (b) $\dfrac{4\pi}{3}$ has a reference angle of $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$.

   (c) $\dfrac{11\pi}{6}$ has a reference angle of $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$.

   Finding the reference angle is a process in which one identifies which quadrant the angle is in, and then applies to correct formula, or logical reasoning. One does not have to **memorize** the formulas so much as understand where they come from. If one can sketch an angle in standard position, then one can find the reference angle.

2. If an angle does not fall in the ranges $(0, 360°), (0, 2\pi)$, then we first find the coterminal angle in those ranges, then we compute the reference angle.

   (a) $\theta = -\dfrac{7\pi}{4}$ is coterminal with $\frac{\pi}{4}$. (We add 2π to θ). As θ is in the first quadrant, θ has a reference angle of $\frac{\pi}{4}$.

   (b) $\theta = \dfrac{21\pi}{4}$. We find the coterminal angle in the range $(0, 2\pi)$. By subtracting 2π twice we obtain $\frac{5\pi}{4}$. Then, θ is in QIII, and so the reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

When one gets more comfortable with angles and finding reference angles, one becomes less reliant on pinpointing the correct formula to use.

10.2.5 Main Take-Away

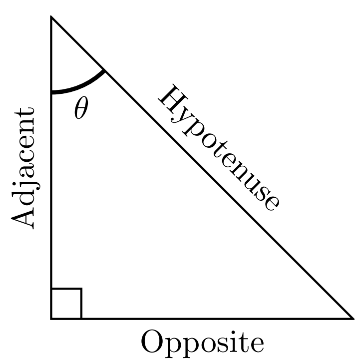
The main take away of this section is to understand what it means to sketch an angle in standard position, what a coterminal angle is, and what a reference angle is. One should become comfortable with both degrees and radians; it can be cumbersome to constantly convert from one to another, especially when under pressure on an exam! Finally, the most important skill one must learn from this section is finding the reference angle of a given angle θ.

1. Check if θ is in the range $(0, 360°)$ if degrees, and $(0, 2\pi)$ if radians.
2. If θ is not in the proper range, then find the coterminal angle of θ that is in the proper range by repeatedly adding or subtracting multiples f $360°$ or 2π.
3. Finally, identify the proper quadrant and calculate the reference angle.

10.3 Trig Functions

Triangles are fundamental in understanding the unit circle, as we will see later on. Triangles have three sides and three angles. In particular, we care about **right triangles**, triangles where one of the three angles is $90°$. Right triangles have three sides, two of which are *legs* and one of which is the *hypotenuse* (the longest side.)

Trigonometric functions are functions which define a relation between a given angle of a right triangle and ratios of the sides of triangles. We have six trig functions, namely sine, cosine, tangent, cosecant, secant, and cotangent. Given a right triangle and one of the non-right angles, we define the sides relative to the angle as 'opposite', 'adjacent' (next to), and the hypotenuse. The abbreviations used should be clear. We define the trig functions as follows.



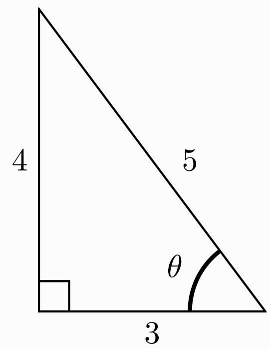
$$\sin(\theta) = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos(\theta) = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan(\theta) = \frac{\text{Opp}}{\text{Adj}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{Adj}}{\text{Opp}}$$

Now one may be given a triangle and a marked angle θ, and then asked to compute the trig function values with θ as an input. Some students prefer to use the mnemonic `SOHCAHTOA` to remember the trig ratios; sine, opposite, hypotenuse, cosine, adjacent, hypotenuse, tangent, opposite, adjacent.

Example – The 3-4-5 Triangle
Consider the following triangle. Remember, trig ratios are calculated relative to θ. Here the side of length 4 is opposite θ, the side of length 3 is adjacent, and the side of length 5 is the hypotenuse (it is opposite the 90° angle.)



$$\sin(\theta) = \frac{4}{5} \qquad \csc(\theta) = \frac{5}{4}$$

$$\cos(\theta) = \frac{3}{5} \qquad \sec(\theta) = \frac{5}{3}$$

$$\tan(\theta) = \frac{4}{3} \qquad \cot(\theta) = \frac{3}{4}$$

Now if one is given a triangle with a missing side length, say the hypotenuse, then we would not be able to calculate $\sin(\theta)$ for a given angle θ. This is where the Pythagorean Theorem comes into play.

Theorem – Pythagorean Theorem

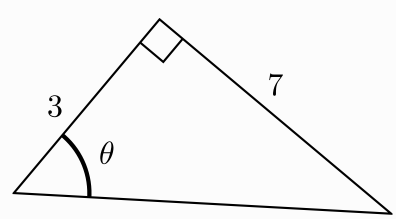
For any right triangle with legs a, b and hypotenuse c, the following is true:

$$a^2 + b^2 = c^2.$$

Indeed, we can verify this using the previous example of a 3-4-5 triangle. The hypotenuse is length 5, and the legs have length 3 and 4. Then

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (3)^2 + (4)^2 &= (5)^2 \\ 9 + 16 &= 25 \quad \checkmark \end{aligned}$$

Example – Missing Hypotenuse Consider the following triangle. Find $\csc(\theta)$.

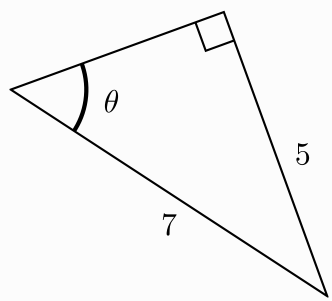


Since $\csc(\theta) = 1/\sin(\theta) = \text{hyp}/\text{opp}$, we must find the hypotenuse. Using the Pythagorean Theorem we have

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (3)^2 + (7)^2 &= c^2 \\ 9 + 49 &= c^2 \\ 58 &= c^2 \\ \pm\sqrt{58} &= c \end{aligned}$$

Here we pick $\sqrt{58} = c$ (the hypotenuse is positive. Soon we'll see that in the xy-plane the hypotenuse is a radius, which is always positive, however the legs may be positive or negative.) From this, we have $\csc(\theta) = \sqrt{58}/7$.

Example – Solve for Missing Leg Consider the following triangle. Find $\sec(\theta)$.



In this triangle, we are given the length of a leg and the hypotenuse. Let $a = 5$ and $c = 7$. Then using the Pythagorean Theorem, we solve for a:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (5)^2 + b^2 &= (7)^2 \\ 25 + b^2 &= 49 \\ b^2 &= 24 \\ b &= \pm\sqrt{24} \\ b &= \pm 2\sqrt{6} \end{aligned}$$

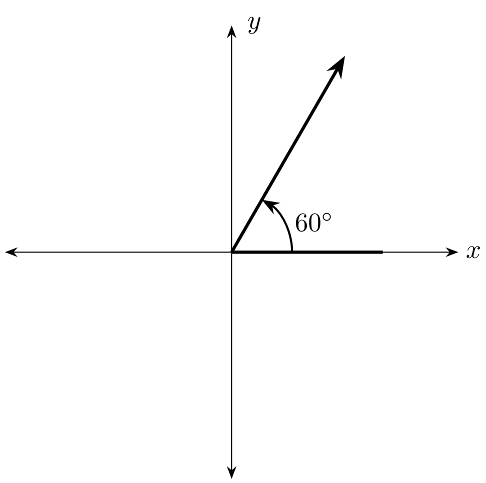
We take $b = 2\sqrt{6}$; as we are not in the xy-plane (there is no "direction" to indicate with the sign of a number), we take the positive version. So $\sec(\theta) = \frac{1}{\cos(\theta)}$, or hypotenuse divided by adjacent. We have

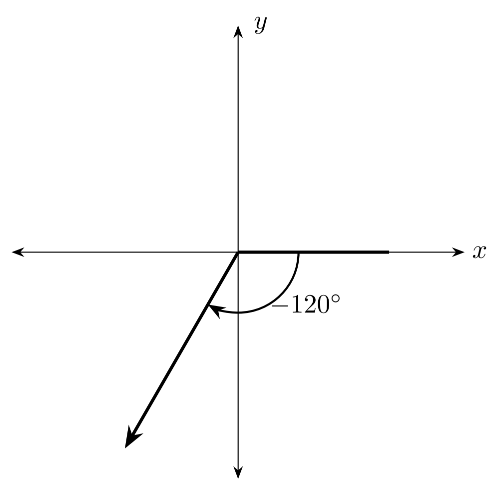
$$\sec(\theta) = \frac{7}{2\sqrt{6}} = \frac{7\sqrt{6}}{12}.$$

Note: we rationalized the denominator by multiplying the numerator and denominator by $\sqrt{6}$.

10.4 The Unit Circle

So far, we have been sketching angles in standard position. That is, we start in the xy-Plane which is divided into four quadrants which can be thought of in terms of the sign of x and y. Namely, QI: $(+,+)$, QII: $(-,+)$, QIII: $(-,-)$, and QIV: $(+,-)$. We then set up a correspondence between angles and lines in the plane; when sketching an angle θ the convention is to start on the positive x axis, and then rotate counter-clockwise or clockwise depending on if θ is positive or negative respectively. We would then draw the **terminal side** of θ. Earlier in these notes, we were sketching angles with an imaginary circle, but what we really should have done is sketch an angle like so:





Notice, when we draw the terminal side of θ, we draw a *ray*, a line starting at the point $(0,0)$ with an arrow indicating that it goes on forever. Each point (x,y) on this line corresponds to the angle θ. This allows us to think of the xy-plane in a manner which is different than before. Previously, we'd think of the plane as two perpendicular axes, x and y, with points (x,y) representing a placement on each axis. This forms a grid. However, another way to think of the xy-plane is to think in terms of angles and a radius. Instead of thinking "to travel to the point (x,y), we move x units horizontally and y units vertically", we can instead think of points in terms of (r,θ): "I rotate θ degrees/radians, then I walk a distance of r."

We now introduce the unit circle. One can define a circle as a set of points which are all the same distance from a specified center point. The *unit circle* is the set of points which are distance 1 from the point $(0,0)$. Each point of the unit circle is some (x,y) in the xy-plane. We can think of these points in terms of the radius and an angle θ: we rotate θ degrees, then travel a distance of 1 unit.

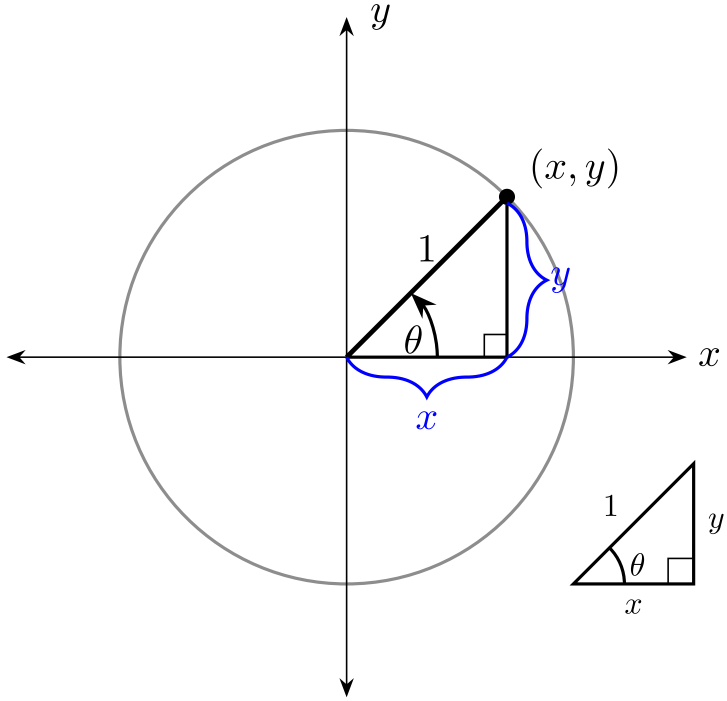


Figure 7: The unit circle. We rotate by an angle of θ, then extend out by a distance of 1 (radius 1). The angle and radius correspond to a point (x, y), and by using right triangles we can describe what x and y are in terms of trigonometric functions.

Notice that for any angle θ, there is a corresponding point (x, y) on the unit circle, and one can draw a right triangle, called a *reference triangle*, with hypotenuse 1, and side lengths x and y (see Figure 7.) Using our knowledge of trigonometric functions, we can see that

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x, \quad \text{and}$$

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y.$$

So, for each point (x, y) on the unit circle, we know $x = \cos(\theta)$ and $y = \sin(\theta)$. Note that $\tan(\theta) = y/x$!

Before we begin to work with the unit circle, we will examine some particular values of θ.

10.4.1 Special Angles and the Unit Circle

There are two special right triangles of interest to us, usually referred to by their angles. We have the 30-60-90 triangle and the 45-45-90 triangle.

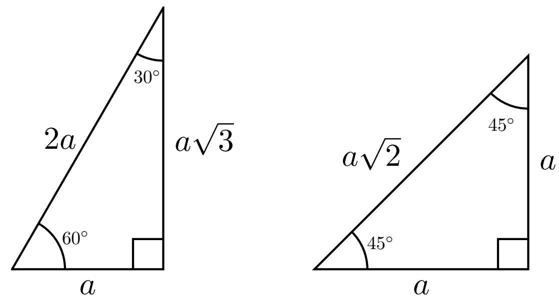


Figure 8: On the left we have the 30-60-90 triangle and on the right we have the 45-45-90 triangle. The variable a is there to show that we can scale these triangles, making each side larger or smaller by the same factor a does not change the ratios of the sides.

In Figure 8 we see the special triangles. Notice the side lengths have a factor of a. We can choose a to be any number; scaling each side by the same factor does not change the angles, or particularly in this context, does not change the **ratios of the sides**. For example, from the 30-60-90 triangle we can calculate

$$\sin(30°) = \frac{a}{2a} = \frac{1}{2}.$$

If we want a hypotenuse of 1 to pretend we are on the unit circle, then we can set $a = 1/2$ for the 30-60-90 triangle and $a = 1/\sqrt{2}$ for the 45-45-90 triangle.

$f(\theta)$ \ θ	$30°$ or $\frac{\pi}{6}$	$45°$ or $\frac{\pi}{4}$	$60°$ or $\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

This table is invaluable to us; these are all the special angles and values we will use. Tip: remembering the table is not as daunting as it seems. First, the rows are sine, cosine, and tangent, and the columns are in order by special angles 30°, 45°, and 60°. The first row, the denominators are all 2, and the numerators are $\sqrt{1} = 1$, $\sqrt{2}$, and $\sqrt{3}$ (ascending order!). Then the second row is the first row in reverse. Finally the third row need not be memorized either; the third row can be computed by computing $\sin(\theta)/\cos(\theta)$. This requires one to be comfortable with dividing fractions (keep-change-flip.) In sum, to remember the table, one needs only remember the first row!

10.4.2 Constructing the Unit Circle with Special Angles

Recall for the unit circle, $x = \cos(\theta)$ and $y = \sin(\theta)$. For our three special angles, we will have three corresponding points $(x, y) = (\cos(\theta), \sin(\theta))$ in the first quadrant. For

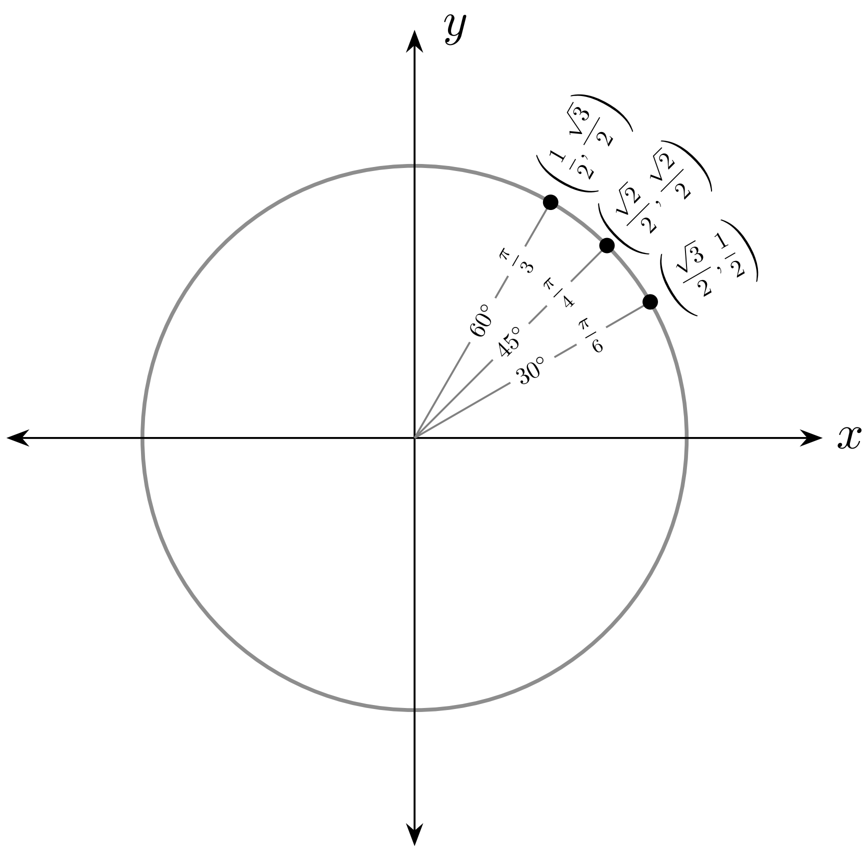


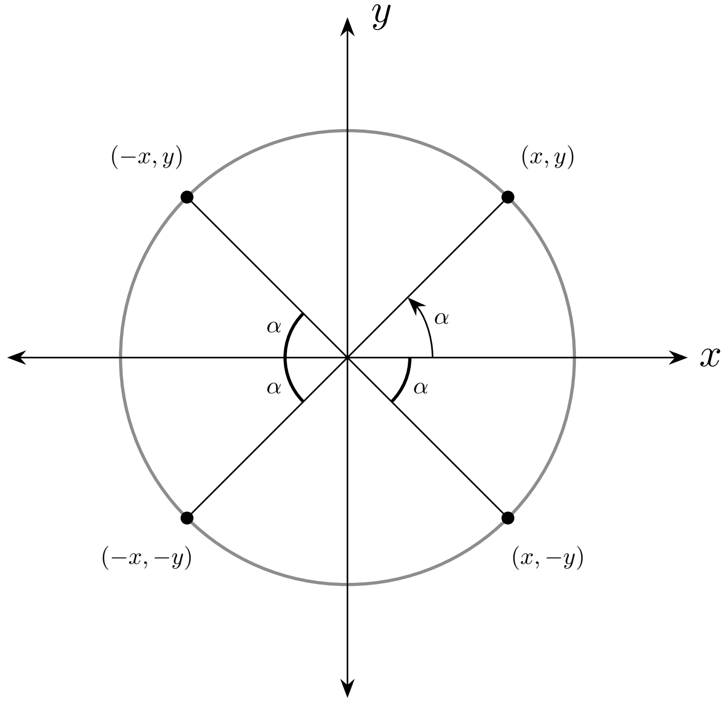
Figure 9: The first quadrant of the unit circle filled with our special angles.

example, the terminal side of 30° intersects the unit circle in the point

$$(\cos(30^\circ), \sin(30^\circ)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Figure 9 shows the three special angles and their corresponding (x, y) point on the unit circle.

What about the points on the unit circle in the second, third, and fourth quadrants? First remember, every point (x, y) in the first quadrant has x and y positive. The beauty of the unit circle is that we only need two things to find the (x, y) point of any θ; the reference angle αp of θ and the corresponding point of α in the first quadrant. That is, any point (x, y) on the unit circle in the first quadrant appears in the other quadrants, up to a change in sign. By "change in sign" we mean that the x or y coordinate may become negative. Recall that the reference angle of an angle θ is the **positive, acute** angle between the terminal side of θ and the x-axis. Every angle has a corresponding reference angle (and since the reference angle is positive and acute, it lives in the first quadrant). Conversely, each angle α in QI corresponds to three other angles, one in each of QII, QIII, and QIV which have α as a reference angle. Note we are focusing on angles which are between 0 and 360° (or 0 and 2π.) In picture form:



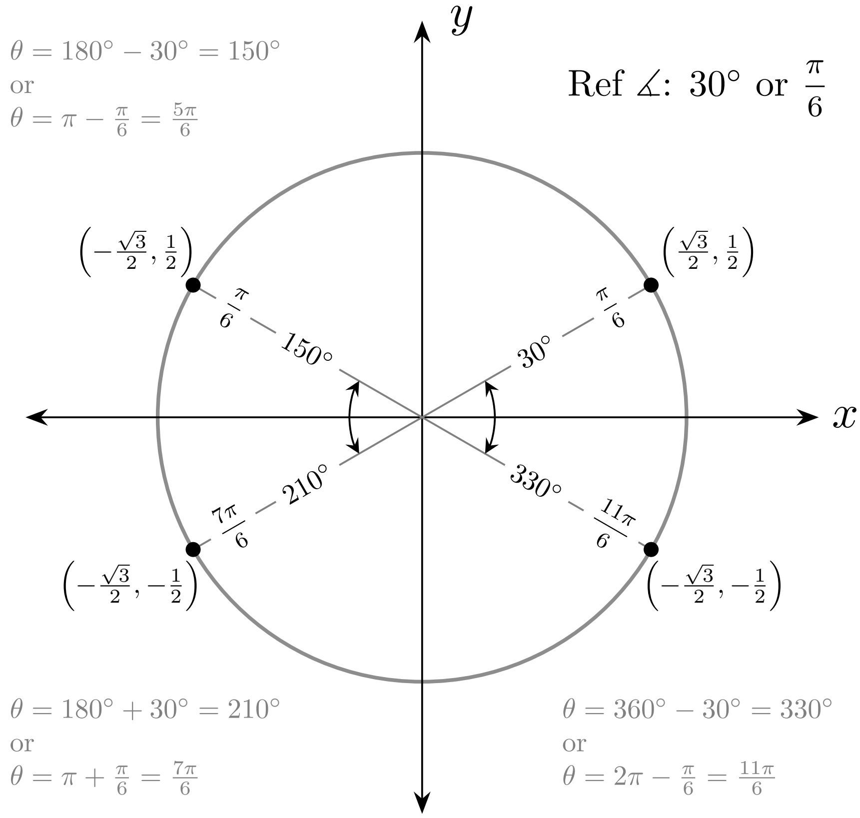
So there are angles with their terminal side in QII, QIII, and QIV with 30°, 45°, and 60° as a reference angle. Let's use 30° to illustrate, and come up with a strategy to remember what these angles are.

To find the angle θ in QII which has a reference angle of 30°, we simply start on the negative x-axis and rotate clockwise 30°. That is, we start at the terminal side of 180°, then rotate 'up' 30°, giving us $\theta = 180° - 30° = 150°$. By using this same logic, for QIII we start at 180° and rotate counter-clockwise 30° to get $\theta = 180° + 30° = 210°$. Finally, in QIV we start at 360° and rotate clockwise by 30° to get $\theta = 360° - 30° = 330°$. Recall previously when learning about reference angles, we were tasked with finding a reference angle given an angle. Now we are doing the reverse.

1. Given an angle θ.
2. Found the quadrant θ is in.
3. Computed the reference angle α of θ given the quadrant.
   (a) QII: $\alpha = 180^\circ - \theta$
   (b) QIII: $\alpha = 180^\circ + \theta$
   (c) QIV: $\alpha = 360^\circ - \theta$

1. Given a reference angle α.
2. Find an angle in QII, QIII, or QIV with α as a reference angle.
   (a) QII: $\theta = 180^\circ - \alpha$
   (b) QIII: $\theta = 180^\circ + \alpha$
   (c) QIV: $\theta = 360^\circ - \alpha$

Of course, one should be comfortable doing so with radians. Once we find all the angles, we adjust the sign of the point in QI corresponding with 30°.



The reader is encouraged to make similar drawings for 45° and 60°. In the end we have a full unit circle in Figure 10. Notice that we did not point out the other special angles 90°, 180°, 270°, and 0° or 360°. Since the unit circle has radius 1, finding the coordinates for these angles should be immediate from the picture (less to memorize.)

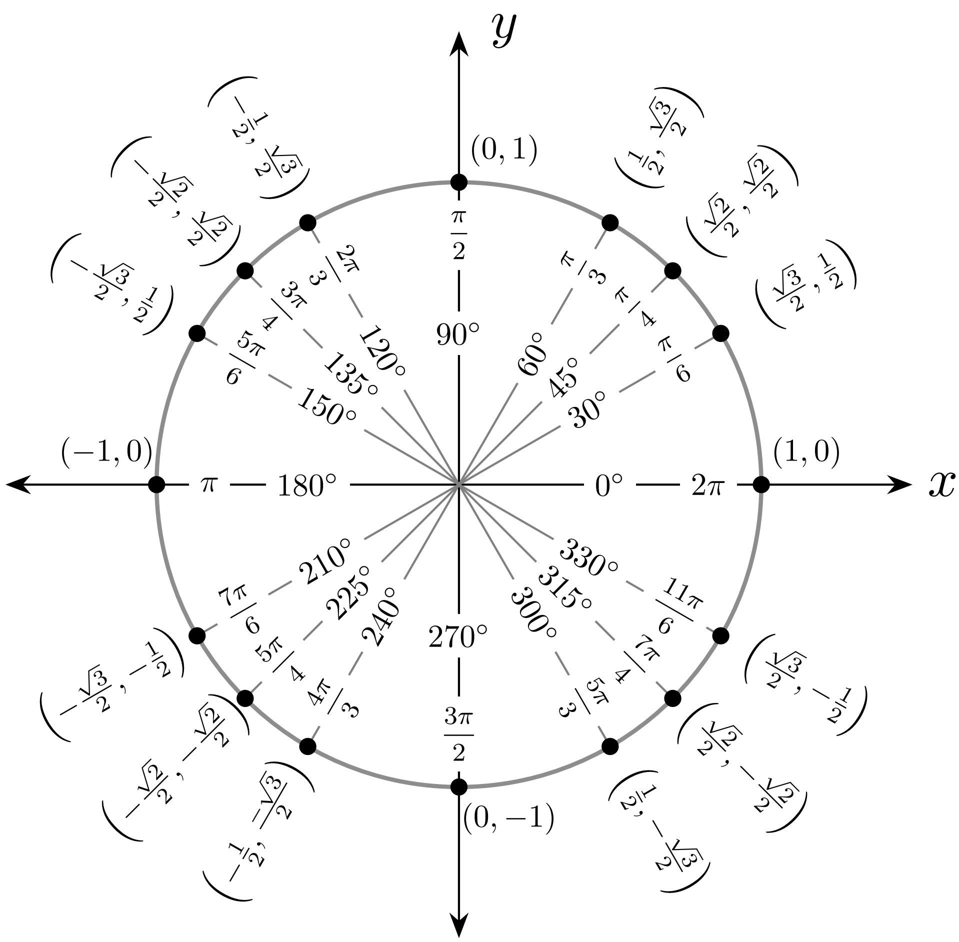


Figure 10: The unit circle filled out. Knowing these values, we can compute sine, cosine, tangent, cosecant, secant, or cotangent of any special angle, or any angle with a special angle as a reference angle.

Example – Computing Trig Functions of Special Angles To practice using the unit circle, here are some values to double check. **In general, first one should check if an angle is in the range 0 to 2π. If not, find the coterminal angle which is in the range, and then compute the reference angle, and make sure the answer has the proper sign.**

1. $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.
2. $\cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.
3. $\sin\left(-\frac{5\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$
4. $\tan(90°)$ is **undefined**, since $\sin(90°) = 1$ and $\cos(90°) = 0$, and $\tan(90°) = \sin(90°)/\cos(90°) = 1/0$.

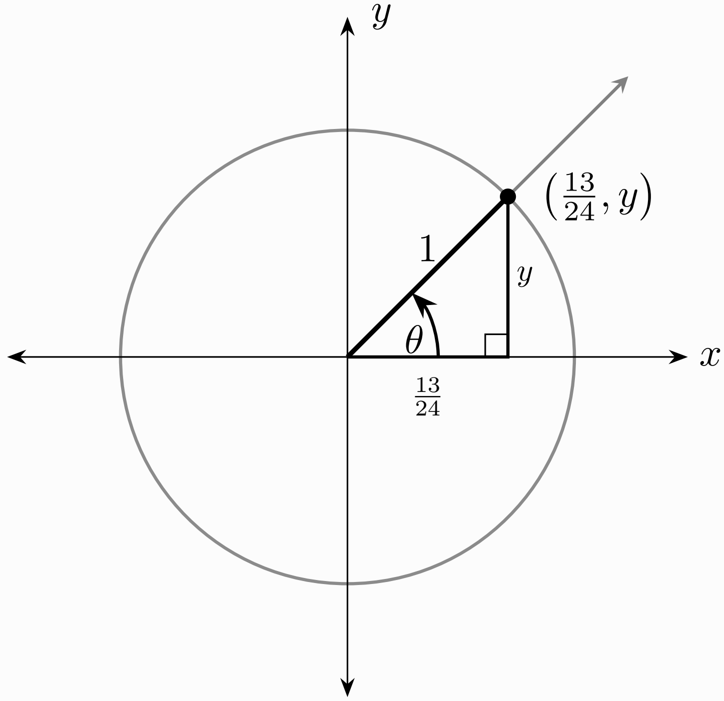
Be careful, as reciprocal trig functions may also be undefined for certain angles, since their denominators may be equal to 0.

10.5 Problems Involving The Unit Circle - Reference Triangles

Terminal Side Intersects Unit Circle

Let θ be an angle. What is the value of $\csc(\theta)$ if the terminal side of θ intersects the unit circle in QI at $x = \frac{13}{24}$?

We can draw a picture to represent this scenario. The terminal side is in QI, and it intersects the unit circle at a point with a specified x coordinate.



Now we have a right triangle with one missing side. Notice the hypotenuse is 1, the radius of the unit circle which is intersected by the terminal side as specified by the problem. To find the missing side, we use the Pythagorean Theorem.

$$\left(\frac{13}{24}\right)^2 + y^2 = 1^2$$

$$\frac{13^2}{24^2} + y^2 = 1$$

$$y^2 = 1 - \frac{13^2}{24^2}$$

$$y^2 = \frac{24^2}{24^2} - \frac{13^2}{24^2}$$

$$y^2 = \frac{24^2 - 13^2}{24^2}$$

$$y^2 = \frac{407}{24^2}$$

$$y = \pm\sqrt{\frac{407}{24^2}}$$

$$y = \pm\frac{\sqrt{407}}{\sqrt{24^2}}$$

$$y = \pm\frac{\sqrt{407}}{24}$$

We take the positive solution, $y = \sqrt{407}/24$ since y is positive in QI. If we were in QIII or QIV, we would choose the negative solution.
We were asked to calculate $\csc(\theta)$.

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y} = \frac{1}{\sqrt{407}/24} = \frac{24}{\sqrt{407}} = \frac{24\sqrt{407}}{407}$$

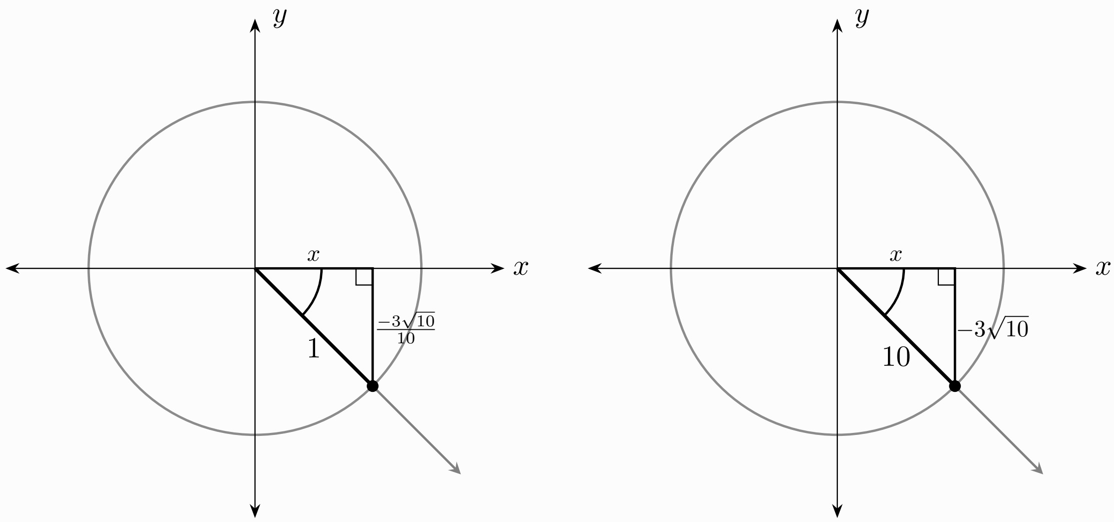
In the previous problem, you may have also used a formula and well known identity, $\cos^2(x) + \sin^2(x) = 1$. This comes from the equation of a circle, $x^2 + y^2 = r^2$, where for the unit circle we substitute $r = 1$. Another way to see this is to look back at Figure 7, notice any reference triangle we draw using a point on the unit circle has sides x and y and hypotenuse 1. Using the Pythagorean Theorem we have

$$x^2 + y^2 = 1 \quad \rightarrow \quad \cos^2(\theta) + \sin^2(\theta) = 1.$$

Example – Arbitrary Radius

Let θ be an angle with terminal side in QIV. Given that $\sin(\theta) = \frac{-3\sqrt{10}}{10}$, what is $\cos(\theta)$.

We can set up a reference triangle in two ways, both yielding the same answer. We can assume that the terminal side of θ intersects the unit circle, or we can instead assume we intersect a circle of larger radius. Since $\sin(\theta) = \frac{-3\sqrt{10}}{10}$, or opposite over hypotenuse when thinking of a reference triangle, we can take the hypotenuse to be 10, and one side to be $-3\sqrt{10}$. Or, we can assume the hypotenuse is 1 and take one side to be $\frac{-3\sqrt{10}}{10}$.



In the case of the left picture, we know on the unit circle $x = \cos(\theta)$, so solving for x will yield our answer. Notice the leg representing the y coordinate is negative in both pictures, since y is negative in QIV. Using the Pythagorean Theorem

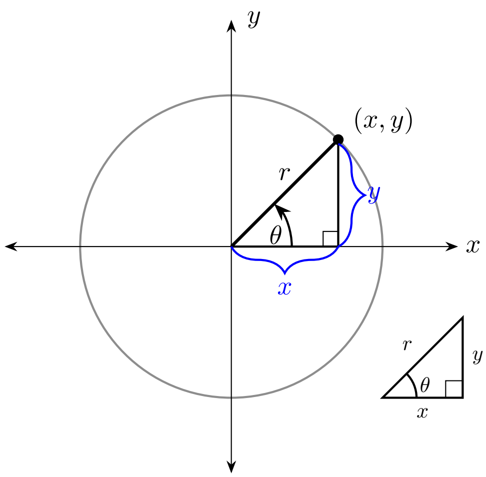
$$\begin{aligned}
x^2 + \left(\frac{-3\sqrt{10}}{10}\right)^2 &= 1^2 \\
x^2 + \frac{90}{100} &= 1 \\
x^2 &= 1 - \frac{90}{100} \\
x^2 &= \frac{10}{100} \\
x^2 &= \frac{1}{10} \\
x &= \pm\sqrt{\frac{1}{10}} \\
x &= \pm\frac{1}{\sqrt{10}}
\end{aligned}$$

Since we are in QIV, x is positive, and so $\cos(\theta) = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$.
If we used the picture on the right, then we would have to solve

$$x^2 + (-3\sqrt{10})^2 = 10^2$$

giving $x = \sqrt{10}$. Then $\cos(\theta)$ is adjacent over hypotenuse, or $\frac{\sqrt{10}}{10}$.

In the previous example, we saw two different set ups for approaching the same problem. We don't always have to assume we are working in the unit circle; the hypotenuse of the reference triangle, or the *radius*, does not have to be 1. We may instead interpret our trig ratios in the following way.



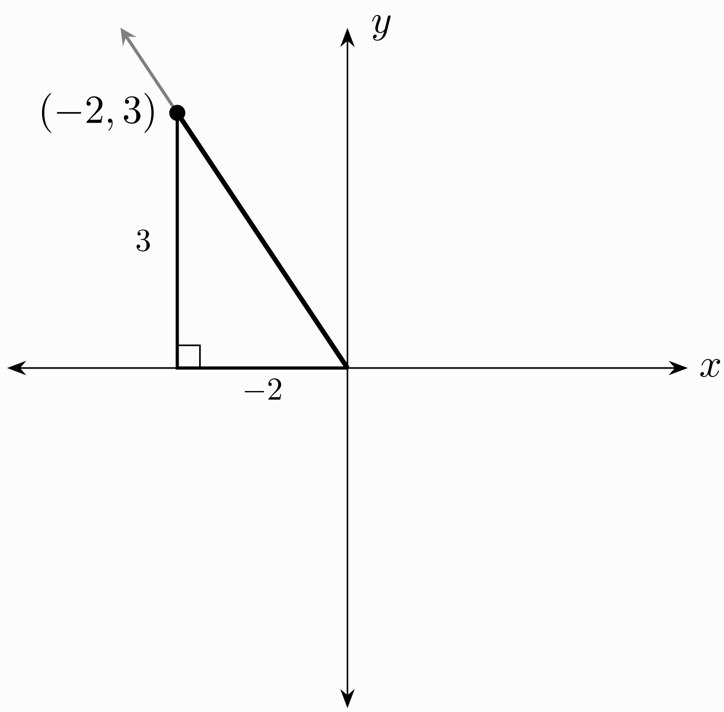
$$\cos(\theta) = \frac{x}{r}$$

$$\sin(\theta) = \frac{y}{r}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}$$

We constrain ourselves to the unit circle often, as trig functions are defined as ratios of sides of right triangles. As the radius grows bigger, the x and y coordinates change, but the ratios between the triangle sides remain the same. However, as we eluded to previously, any point in the plane corresponds to a radius and an angle. Specifically, we can rotate and then walk in the direction of the terminal side to get to any point, and this distance is not always 1 (that is, radius is not always 1) as we'll see in the next example.

Example – Given Point
Let $(-3, 2)$ be a point on the terminal side of an angle θ. Find $\sin(\theta)$, $\sec(\theta)$, and $\tan(\theta)$.
The point $(-3, 2)$ is in quadrant 2, so the terminal side of θ is in QII. We can draw a reference triangle.



To evaluate the trig functions given, we need to solve for the hypotenuse of the reference triangle. We have

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-2)^2 + (3)^2 &= r^2 \\ 4 + 9 &= r^2 \\ 13 &= r^2 \\ \pm\sqrt{13} &= r \end{aligned}$$

The radius, or hypotenuse is always positive so we take $r = \sqrt{13}$. Now we have

$$\sin(\theta) = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\sec(\theta) = \frac{\sqrt{13}}{-2}$$

$$\tan(\theta) = \frac{3}{-2}$$

Notice that we needed to solve for the hypotenuse to calculate $\sin(\theta)$ and $\sec(\theta)$, but since $\tan(\theta) = y/x$ we could have evaluated that right away.

Example – Extra Example

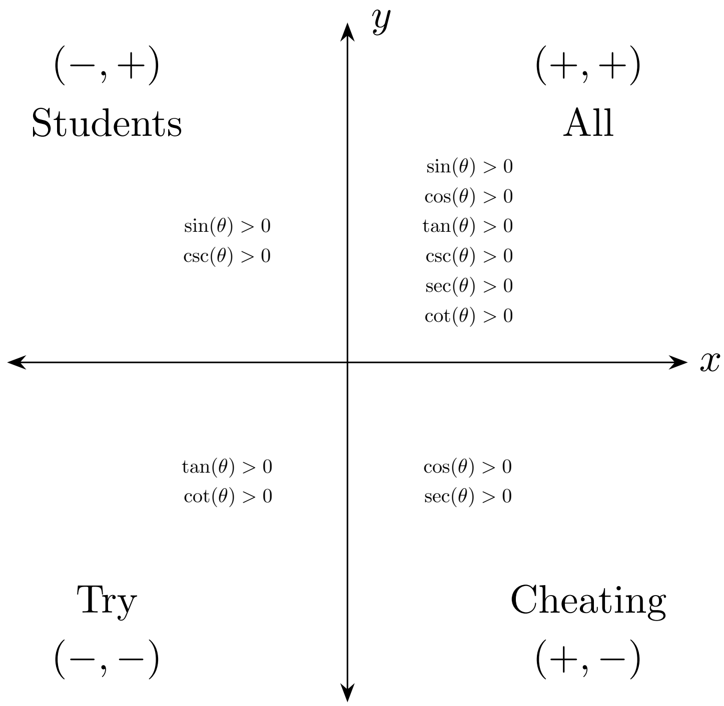
Let θ be an angle such that $\cos(\theta) = \frac{7}{8}$ and $\tan(\theta) > 0$. Find $\cot(\theta)$ and $\sin(\theta)$. Here we are not given the quadrant that θ is in, however we can use the information given to determine the quadrant. Since $\tan(\theta) > 0$, this implies that $y/x > 0$. In other words, either x and y are both positive, or x and y are both negative. That is, we are either in QI or QIII. Since $\cos(\theta) = 7/8 = x/r$, which is positive, and since r is always positive, it must be that x is positive. Since x is negative in QIII, we must be in QI.

Now we have θ in QI with $\cos(\theta) = 7/8$. Draw the reference triangle to show that $\cot(\theta) = 7\sqrt{15}/15$ and $\sin(\theta) = \sqrt{15}/8$.

In this last example we were not given a quadrant, and instead had to use context clues to figure out what quadrant the terminal side of θ lies in.

10.6 Identifying Quadrant From Trig Function Sign

Given information about two trig functions with regard to θ, we can identify which quadrant the terminal side of θ is in. As we saw before, $\cos(\theta) = x/r$, $\sin(\theta) = y/r$, and $\tan(\theta) = y/x$ (remember the reciprocals secant, cosecant, and tangent.) since r is always positive, if we know whether or not a trig function is positive or negative, we can deduce the sign of x and y. It may help some students to commit saying "**A**ll **S**tudents **T**ry **C**heating" to memory.



Let's look at some examples.

Example – Determine the Quadrant Determine the quadrant in which the terminal side of θ lies if

1. $\cos(\theta) > 0$ and $\sin(\theta) > 0$.

   Here, $\cos(\theta > 0$ tells us that we are either in QI or QIV. Out of these, $\sin(\theta)$ is positive in QI only. Therefore θ terminates in QI.

2. $\cot(\theta) < 0$ and $\cos(\theta) < 0$.

   Here, $\cot(\theta) < 0$ tells us we are either in QII or QIV (cot follows tan, the reciprocal). Then $\cos(\theta) < 0$ tells us θ terminates in QII.

Here the process is simple and step by step; find the possible quadrants for each trig function and see which quadrant they have in common.

10.7 The Graph of Sine and Cosine

- Amplitude, Frequency/Period, Phase Shift, Vertical Shift, then all together.
- Summary of what a function with all the features looks like, with example.

10.7.1 Sketching Graphs – Complete Examples

Example – Sketching sine

Sketch the graph of the function $y = 2\sin\left(3x - \frac{\pi}{4}\right) + 1$.

First, we write our function in general form:

$$y = 2\sin\left(3x - \frac{\pi}{4}\right) + 1 = 2\sin\left(3\left(x - \frac{\pi}{12}\right)\right) + 1.$$

Let's list some info:

1. $a = 2$, so we have an **amplitude** of $|2| = 2$.
2. $b = 3$, so we have a **period** of $\frac{2\pi}{3}$.
3. A **phase shift** of $\frac{\pi}{12}$ (right). One can find the phase shift by either factoring out b, or setting the inside of the sine function equal to 0 and solving.

$$\begin{aligned} 3x - \frac{\pi}{4} &= 0 \\ 3x &= \frac{\pi}{4} \\ x &= \frac{\pi}{12} \end{aligned}$$

   This is both the **starting point** of the primary cycle and the phase shift (in this case, $\pi/12$ units to the right.)

4. Vertical shift $d = 1$, so consequently the midline equation is $y = 1$.

We need to find the new x values for the five points of the primary cycle that we will sketch.

Finding New x Values.

Method 1 – Find Δx.

First, we solve for the first critical x value. This is the start of our cycle. We have

$$3x - \frac{\pi}{4} = 0 \to x = \frac{\pi}{12}.$$

In $\sin(x)$ function, our original five points values are $\pi/2$ apart in x.

x	y
0	0
$\pi/2$	1
π	0
$3\pi/2$	-1
2π	0

How far apart are our critical values for $2\sin\left(3x - \frac{\pi}{4}\right) + 1$? It all comes from b. The phase shift of $\pi/12$ tells us where to start, at $x = \pi/12$.
All we do to find the new distance between critical values is calculate $\Delta x = \pi/2b$. This can be explained in two ways:

1. The original gap between the points is $\pi/2$. Divide each gap size $\pi/2$ by b, or in other words $\Delta x = \dfrac{\pi}{2b}$.
2. The length of one cycle is the **period**, $2\pi/b$. There are five original points, and so there are **four consecutive gaps** between these points. So if we divide the length of the cycle by four, we get the length of each gap. So $\dfrac{\frac{2\pi}{b}}{4} = \dfrac{\pi}{2b}$.

Some students may find it easier to remember $p/4$; calculating the period and then dividing that by 4.
Now from $\pi/12$, the value $b = 3$ tells us that the next critical value is

$$\frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$$

away. So consecutive points are $\pi/6$ apart.

$$x = \frac{\pi}{12} \qquad (x\text{-int})$$

$$x = \frac{\pi}{12} + \frac{\pi}{6} = \frac{3\pi}{12} \qquad (\text{Max})$$

$$x = \frac{3\pi}{12} + \frac{\pi}{6} = \frac{5\pi}{12} \qquad (x\text{-int})$$

$$x = \frac{5\pi}{12} + \frac{\pi}{6} = \frac{7\pi}{12} \qquad (\text{Min})$$

$$x = \frac{7\pi}{12} + \frac{\pi}{6} = \frac{9\pi}{12} = \frac{3\pi}{4} \qquad (x\text{-int})$$

Method 2 – Five Equations

We can set up five equations to find the new x values of our five special points:

$$3x - \frac{\pi}{4} = 0 \qquad \rightarrow \qquad x = \frac{\pi}{12}$$

$$3x - \frac{\pi}{4} = \pi/2 \qquad \rightarrow \qquad x = \frac{\pi}{4}$$

$$3x - \frac{\pi}{4} = \pi \qquad \rightarrow \qquad x = \frac{5\pi}{12}$$

$$3x - \frac{\pi}{4} = 3\pi/2 \qquad \rightarrow \qquad x = \frac{7\pi}{12}$$

$$3x - \frac{\pi}{4} = 2\pi \qquad \rightarrow \qquad x = \frac{3\pi}{4}$$

To obtain the new y-values, we take our primary y values, multiply by a and add d. Then we have a complete table of new values.

x	y
0	1
$\pi/2$	3
π	1
$3\pi/2$	-1
2π	1

Sanity check: We calculated the period to be $\frac{2\pi}{3}$. This is the length of one cycle. If we take the difference between our starting and end point, we should get the period.

$$\frac{9\pi}{12} - \frac{\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}.$$

We know the amplitude is 2, so we can check for that too: $|a| = \frac{\max - \min}{2} = 2$.

Formulas to remember:

$$|a| = \frac{\max - \min}{2}$$
$$d = \frac{\max + \min}{2}$$
$$p = \frac{2\pi}{b}$$
$$\Delta x = p/4$$

10.8 The Graph of Secant, Cosecant, Tangent, and Cotangent.

11 Trig Inverses

$\sin^{-1}(x)$ and $\tan^{-1}(x)$ use the right side of the unit circle, while $\cos^{-1}(x)$ uses the upper half.

12 Trig Equations