Prerequisites

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Math 151 - 901

Contents

0 Prerequisites

0.1 Exponent Rules

Exponents are a convenient notation in mathematics. They are used to represent the multiplication of the same number some number of times. For example, instead of writing $5 \times 5 \times 5$, we may write 5^3 to communicate to the reader: "multiply 5 by itself 3 times". Not only does this make reading mathematics simpler, but it also allows us to more easily manipulate algebraic expressions. For example, we use these concepts to expand expressions like $(x+5)^2$ or simplify expressions like $\frac{(x+5)^2(x-2)}{(x+7)(x+5)^{10}}$.

Here a, b, p, q are any real number. Verify each property using your own numbers.

Name	Form	Example
Zero Rule	$a^0 = 1$	$2^0 = 1$
Product Rule	$a^p \cdot a^q = a^{p+q}$	$2^10 \cdot 2^4 = 2^{14}$
Quotient Rule	$a^p / a^q = a^{p-q}$	$2^{10}/2^4 = 2^6$
Power of a Product	$(a \cdot b)^p = a^p b^p$	$(2 \cdot 5)^2 = 2^2 \cdot 5^2$
Power of a Quotient	$(a/b)^p = a^p/b^p$	$(5/4)^3 = 5^3/4^3$
Power of a Power	$(a^p)^q = a^{pq}$	$(2^3)^2 = 2^6$
Negative Exponent	$a^{-p} = 1/a^p$	$2^{-1} = 1/2$
Fractional Exponent	$a^{p/q} = (\sqrt[q]{a})^p = (\sqrt[q]{a^p})$	$5^{4/3} = \sqrt[3]{5^4}$

Most often we will use a combination of these rules to simplify algebraic expressions. **Note**: when simplifying an expression, there may be more than one way to apply a sequence of rules. In other words, there is no set order, so long as the rule is implemented properly from one expression to another.

Example. Simplifying Expressions With Exponents. Simplify.

$$\left(\frac{15a^3b^{-5}c^{-2}}{5a^{-7}b^8}\right)^{-2}$$

$$\left(\frac{15a^3b^{-5}c^{-2}}{5a^{-7}b^8}\right)^{-2} = \left(\frac{3a^3b^{-5}c^{-2}}{a^{-7}b^8}\right)^{-2} = \frac{3^{-2}a^{-6}b^{10}c^4}{a^{14}b^{-16}} = \frac{b^{10}b^{16}c^4}{3^2a^6a^{14}} = \frac{b^{26}c^4}{9a^{20}}$$

Steps:

1. We simplify the $\frac{15}{5}$ to 3 in the numerator.
2. Using the **Power of a quotient** and **Power of a product** rules, we "distribute" the exponent -2 to each term on the inside.
3. Using the **Negative Exponent Rule**, we take variables with negative exponents from the top and move them to the bottom while making their exponents positive, and from the bottom to the top while making their exponents positive.
4. Using the **Product Rule**, we can combine identical variables and their exponents. E.g. $a^3a^7 = a^{3+7} = a^{10}$.

0.2 Factoring

Similar to how one uses exponent rules, factoring is a technique we can use to simplify expressions. For example, the fraction $\frac{36}{8}$. One may deduce that 36 and 8 have common factors, and so if we take the *greatest common factor*, in this case 4, we can simplify:

$$\frac{36}{8} = \frac{4 \cdot 9}{4 \cdot 2} = \frac{9}{2}.$$

Similarly for expressions with variables, we may do the same:

$$\frac{4x^2 + 2x}{2x^4 + 4x^2} = \frac{2x(2x+1)}{2x^2(x^2+2)} = \frac{2x+1}{x(x^2+2)}.$$

Factoring allows us to do many things; by writing a quadratic function (and later, polynomial functions in general) as a product of its factors, we can more easily find things like the x-intercept. For example

$$f(x) = x^2 + 5x + 6 = (x+2)(x+3).$$

Multiply the factors out and check this for yourself. In factored form we see that plugging in -2 or -3 results in an evaluation of 0, i.e. $f(-2) = f(-3) = 0$.

0.2.1 Quadratics

We will explore the ways to factor a quadratic expression $ax^2 + bx + c$ with integer coefficients. **If you simply need a quick refresher, look over the examples**.

If $a = 1$, then we have $x^2 + bx + c$. We want to write

$$x^2 + bx + c = (x+p)(x+q).$$

If we multiply out the right hand side:

$$\begin{aligned}(x+p)(x+q) &= x^2 + qx + px + pq \\ &= x^2 + (p+q)x + pq \\ &= x^2 + bx + c.\end{aligned}$$

So, if we equate the coefficients we must find two numbers p and q so that

$$\begin{gathered} p + q = b, \text{and} \\ pq = c. \end{gathered}$$

To do this, we construct a list of factors of c, and check the sums. Let's do an example.

Example. Factoring $x^2 + bx + c$.
Factor $x^2 + 3x - 18$ completely.
Solution. We want two numbers that sum to 3 and multiply out to -18. We can factor 18 as: 1×18, 2×9, or 3×6.
Since what we actually need to factor is -18, **exactly one** of the factors in the pair we choose must be negative, and one must be positive. The choice in this case is $(-3, 6)$, since $-3 + 6 = 3$ and $(-3)(6) = -18$. We get $(x - 3)(x + 6)$ as our answer.

If $a \neq 1$, then we have $ax^2 + bx + c$. Now in this case if we wish to factor, the leading terms of the factors must have a coefficient which is not 1.

We can use the "ac-method" here, and *grouping*. First, we find the factors of ac. Then we find a pair of factors of ac which sum to b. Then, we split the linear term bx into two terms and group.

Example. $ax^2 + bx + c$ using the ac-method.
Factor completely:

$$3x^2 + 13x + 10$$

Solution. Using the ac-method: We have $a = 3, c = 10$, and so $ac = 30$. We want two numbers that add up to $b = 13$ and multiply to $ac = 30$. The number 30 can be factored as:

30	
1	30
2	15
3	10
5	6

We pick 3 and 10:

$$\begin{aligned} 3x^2 + 13x + 10 &= \overbrace{3x^2 + 3x}^{3x} \overbrace{+10x + 10}^{10} \\ &= 3x(x + 1) + 10(x + 1) \\ &= (x + 1)(3x + 10) \end{aligned}$$

Finally, one can use the quadratic formula to find the roots of a quadratic, and then write the quadratic in standard form. The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula will give two roots α_1 and α_2 (sometimes they are the same, this happens when the original quadratic is a perfect square.) Then we write $f(x)$ as $(x - \alpha_1)$ and $(x - \alpha_2)$.

Example. Using the quadratic formula.
Let's look at our previous two quadratic expressions.
For $x^2 + 3x - 18$ we have $a = 1$, $b = 3$, and $c = -18$. Plugging into the quadratic formula we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-18)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{81}}{2} \\ &= \frac{-3 \pm 9}{2}. \end{aligned}$$

So one root $\alpha_1 = \frac{-3+9}{2} = 3$ and the other root $\alpha_2 = \frac{-3-9}{2} = -6$. Finally to write our quadratic in factored form, we have

$$x^2 + 3x - 18 = (x - \alpha_1)(x - \alpha_2) = (x - 3)(x - (-6)).$$

One can also find each factor by solving two equations.

$$\begin{gathered} x = 3 \text{ and } x = -6 \\ x - 3 = 0 \text{ and } x + 6 = 0 \end{gathered}$$

For $3x^2 + 13x + 10$ we have $a = 3$, $b = 13$, and $c = 10$. Plugging in we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(13) \pm \sqrt{(13)^2 - 4(3)(10)}}{2(3)} \\ &= \frac{-13 \pm \sqrt{49}}{6} \\ &= \frac{-13 \pm 7}{6} \end{aligned}$$

So one root $\alpha_1 = \frac{-13+7}{6} = -1$ and the other root $\alpha_2 = \frac{-13-7}{6} = \frac{-10}{3}$.
So the roots are $x = -1$ and $x = \frac{-10}{3}$. Solving each, we get factors $(x + 1)$ and $(3x + 10)$.

0.3 Functions

0.3.1 What is a function?

Recall what a function is:

Definition (Function)

A *function* is an assignment of the elements of one set to another; a function is an object that takes inputs and produces exactly one output. For example, $f(x) = x + 5$. The f is the *name* of the function, the input is x, and $f(x)$ is read "f of x". The *output* $f(x)$ is $x + 5$, or 5 more than the input. Notice, any input for $f(x)$ receives one, and only one, output. The set of inputs can be all real numbers (and consequently the outputs are real numbers.)

Functions can be represented using formulas, or as a picture on a grid.

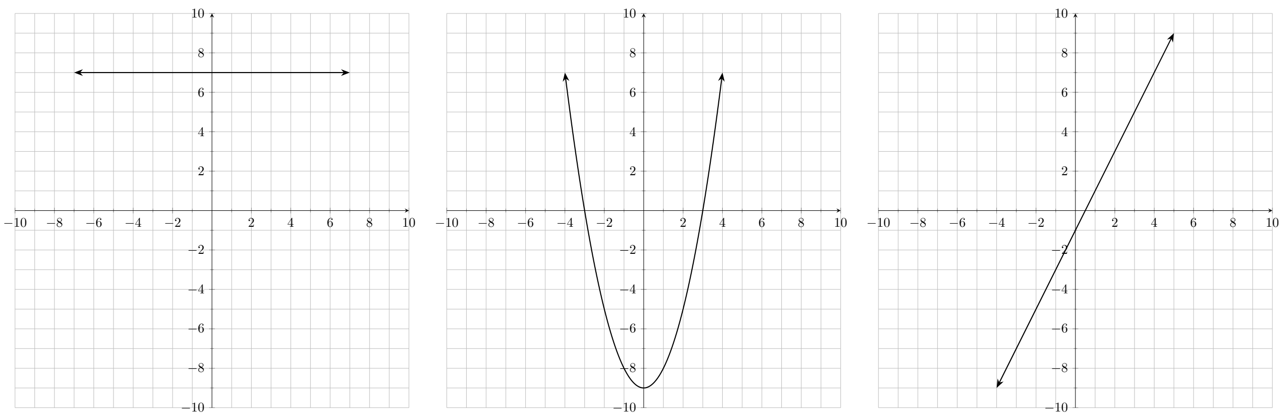

Figure 1: Three functions, from left to right: $f(x) = 7$ (a constant function), $g(x) = x^2 - 9$ (a quadratic function), and $h(x) = 2x - 1$.

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Note how each of the functions in Figure 1 passes the *vertical line test*; any vertical line drawn on the grid intersects the function in exactly one point. This tests the requirement that a function has exactly one output for each of its inputs (you plug one number in, you get one number out.)

The set of allowable inputs of a function is its *domain*, and the set of possible outputs given those inputs is its *range.* The domain and range can be expressed using *interval notation.*

Example. Functions – Domains and Ranges. Taking a look at Figure 1, we can make the following table

Function	Domain	Range
$f(x)$	$(-\infty, \infty)$	$\{7\}$
$g(x)$	$(-\infty, \infty)$	$[-9, \infty)$
$h(x)$	$(-\infty, \infty)$	$(-\infty, \infty)$

The range of f only contains 7, since it is the constant function $f(x) = 7$. The range of g is $[-9, \infty)$; g is a quadratic function with a positive leading coefficient, so it has a *minimum*, in this case -9. Finally, $h(x) = 2x - 1$ is a line, with unrestricted domain and range. Notice that none of these examples has a restricted domain, but there is a function we all know of that can serve as an example. Consider $p(x) = \frac{1}{x}$. Since $\frac{1}{0}$ is undefined, the function p has no output for $x = 0$, that is, 0 is not in the domain of p. This is the only *excluded value* of p, and so the domain of p can be written as $(-\infty, 0) \cup (0, \infty)$. We will discuss functions like p when we talk about rational functions.

We will learn many different classes of functions in addition to the ones we know already (lines, quadratic.) For each family of functions, it is important to know how the domain and range behave.

You should be familiar with *lines* and *quadratic functions.* Recall that x-intercepts are where a function crosses the x axis (when $f(x) = 0$), and the y-intercept is where a function crosses the y-axis (the point $(0, f(0))$).

0.3.2 Lines and Quadratic Functions

Definition (Line)

A *line* is a function of the form $f(x) = mx + b$, where x is the input, m is the *slope*, and b is the y-intercept.
Given two points, one is able to define the unique line passing through the two points. To calculate the slope between two points $(x_1, y_1), (x_2, y_2)$ we use the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Slope is often referred to as *rise over run*, and this can be seen in the formula. The "rise" is the difference between the y coordinates of the points, $y_2 - y_1$, and the "run" is the difference between the x coordinates, $x_2 - x_1$. Notice that one can compute $y_2 - y_1$ or $y_1 - y_2$, so long as one remains consistent. In other words, take care to not to compute something like $\frac{y_2 - y_1}{x_1 - x_2}$.
The *domain* of any line is $(-\infty, \infty)$. The *range* of a line with slope $m \neq 0$ is $(-\infty, \infty)$. If the slope $m = 0$, then the line is *horizontal* with equation $f(x) = b$, and thus has range $\{b\}$.

Let's do an example.

Example - Find Equation Given Two Points

Find the equation of the line passing through the points $(-2, 1)$ and $(1, 7)$.

Solution. A line has equation $f(x) = mx + b$. We must find m and b. Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (1, 7)$.
To find m we calculate the rise over run:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 1}{1 - (-2)} = \frac{6}{3} = 2.$$

Now there are two ways we can obtain the equation. First, we can solve $f(x) = mx + b$ using one of the given points. For example using $(1, 7)$:

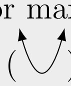
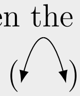
$$7 = (2)(1) + b \implies b = 5.$$

One can also use **point-slope form.** This is essentially the same. Let's use the other point here.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (1) &= 2(x - (-2)) \\ y - 1 &= 2(x + 2) \\ y - 1 &= 2x + 4 \\ y &= 2x + 5 \end{aligned}$$

Given pieces of information of a line, e.g. two points, or one point and the slope, one should be able to find the equation of the line.

Definition (Quadratic Function)

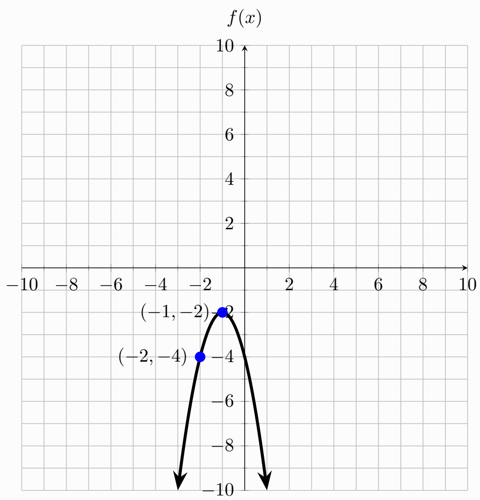
A *quadratic* function is of the form $f(x) = ax^2 + bx + c$ where a, b, and c are *coefficients*, with c referred to specially as the *constant* term. The *vertex* (h, k) of a quadratic function is the minimum or maximum point of the function. If a is positive, then the vertex is a minimum (). If a is negative, then the vertex is a maximum ().
Every quadratic function can be written in what is called *vertex form* $f(x) = a(x - h)^2 + k$. The former is *standard form.*
The domain of any quadratic function is $(-\infty, \infty)$. The range of a quadratic

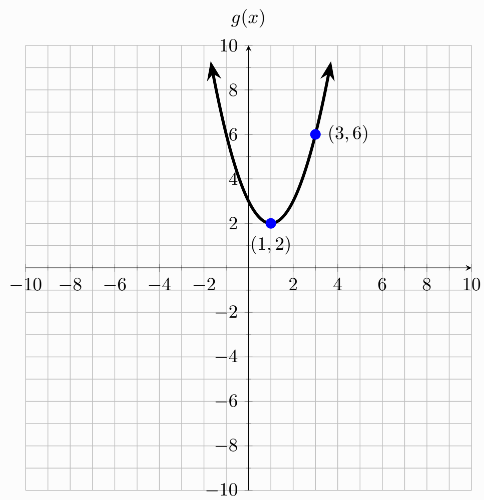
function depends on whether a is negative or positive. For a vertex (h, k), the range of a quadratic function is

1. $[k, \infty)$ if $a > 0$ (positive)
2. $(-\infty, k]$ if $a < 0$ (negative)

Example. A quadratic function

Let's take a look at the following two quadratic functions.

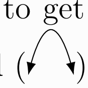




On the left we have a quadratic function $f(x)$ with vertex $(h, k) = (-1, -2)$. To find the equation of f, we can use the vertex form. So $f(x) = a(x - h)^2 + k$. Plugging in we get $f(x) = a(x - (-1))^2 + (-2)$, or $f(x) = a(x + 1)^2 - 2$. To find a, we can plug in a point *other than the vertex* to solve for a. If we plug in the vertex, then $a(x + 1)^2$ will become 0, leaving you unable to solve for 0. Let's use the point $(-2, -4)$ given to us instead. Then

$$\begin{aligned} f(-2) &= a(-2 + 1)^2 - 2 \\ &= a - 2 \end{aligned}$$

and since we know $f(-2) = -4$, we can solve $a - 2 = -4$ to get that $a = -2$.

For a quick mental check, notice how f opens up downward (); this matches with the fact that the a we found is negative. So, $f(x) = -2(x + 1)^2 - 2$. The domain of f is $(-\infty, \infty)$ and the range of f is $(-\infty, -2]$.

Try and follow the same procedure to find that $g(x) = (x - 1)^2 + 2$. The domain of g is $(-\infty, \infty)$ and the range is $[2, \infty)$.

If one is only given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, then one can still find information, such as the vertex, in different ways. If the vertex is (h, k), then an equation for h is

$$h = \frac{-b}{2a}.$$

Once h is calculated, one can plug h into f to get k. The a in the standard form is the same as the a in the vertex form.

Given the equation of a quadratic function that is not in standard form, we can take steps to rewrite it in vertex form. One way is by completing the square.

Example. Completing the square.
Let $f(x) = 3x^2 + 12x + 6$. First, factor the GCF so that $f(x) = 3(x^2 + 4x + 2)$. We complete the square with the inside function by first computing $(b/2)^2$. In this case $(b/2)^2 = (4/2)^2 = 4$. Then we add and subtract $(b/2)^2$. This is like adding 0, but it provides us with a convenience!

$$\begin{aligned} f(x) &= 3(x^2 + 4x + 4 + 2 - 4) \\ &= 3(x^2 + 4x + 4 - 2) \\ &= 3(x^2 + 4x + 4) - 6 && \text{(Move } -2 \text{ out.)} \\ &= 3(x + 2)^2 - 6 \end{aligned}$$

So we see that f is a quadratic function with vertex $(-2, -6)$.

Completing the square can be useful for solving algebra equations involving radicals, which we will see later.

0.3.3 Function Translations

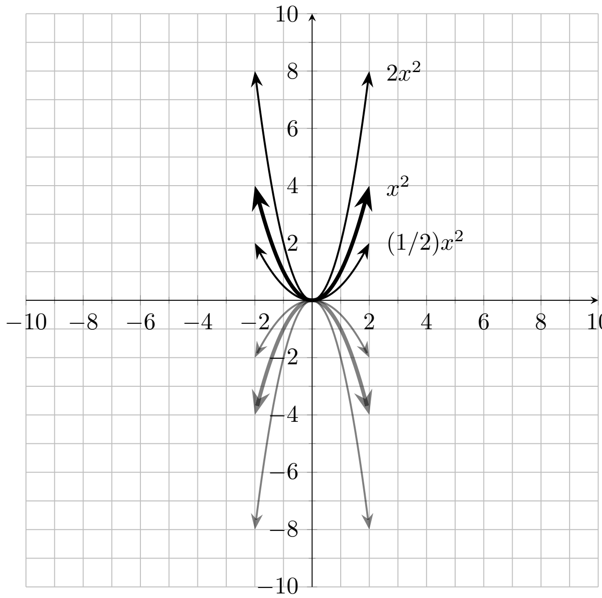
Recall that functions can be stretched, compressed, shifted left to right, and shifted up and down. These concepts are crucial when understanding families of functions, and how to get from a standard example (for example, $f(x) = x^2$, to something like $f(x) = (x + 5)^2 + 2$.

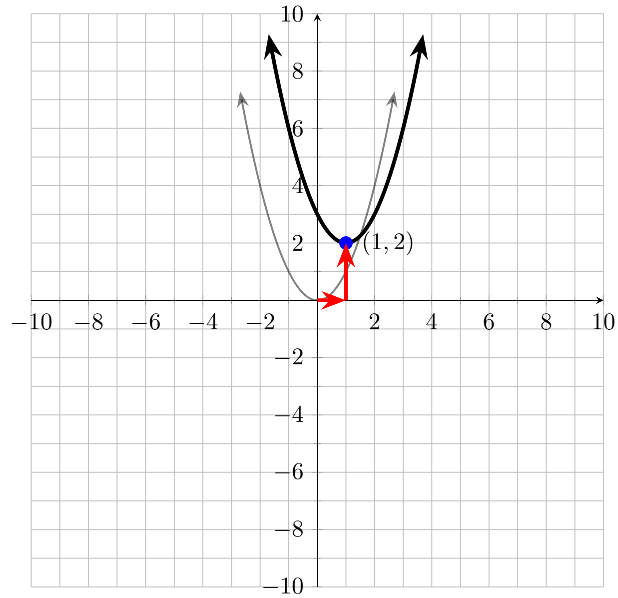
Let f be a function, and let a be any **positive** number. Then we have the following rules.

- $f(x - a)$ shifts a function a units to the right.
- $f(x + a)$ shifts a function a units to the left.
- $f(x) + a$ shifts a function a units up.
- $f(x) - a$ shifts a function a units down.

- If $a > 1$ then $af(x)$ is $f(x)$ but compressed horizontally. If a is a proper fraction (between 0 and 1) then $af(x)$ is $f(x)$ stretched horizontally. This rule applies the same for when a is negative, except the negativity of a reflects $f(x)$ across the x-axis.

To see the shifting in action, go back to the examples of quadratic functions. Notice that vertex form of a quadratic function shows you exactly what shifting is going on. For example, $g(x) = (x-1)^2 + 2$ is the function $f(x) = x^2$ shifted to the right 1 unit, and up 2 units. In other words, $g(x) = f(x-1) + 2$.





You can play with the following Desmos activity to see how the shifting occurs: https://www.desmos.com/calculator/mmkk5legqy.

Summary

List of things you need to know.

- Exponent rules.
- What are the different factoring techniques.
  - Factor $ax^2 + bx + c$ when $a = 1$ and when $a \neq 1$.
- What is a function? How to work with functions (their names, inputs, outputs, etc.) What is the vertical line test?
- What is the domain and range of a line? A quadratic function?
- How do you find the equation of a line given some information?
- How do you find the equation of a quadratic function given some information?
  - How do you find the vertex ($h = \frac{-b}{2a}$).
  - How do you convert a quadratic function written in standard form to vertex form?
    - Completing the square.
    - OR finding the vertex, then find another point of the function and use that to solve for a in the vertex form.
- Understand function translations.