Chapter 9 – Population Growth Models

Notes on Linear and Exponential Sequences

Rayan Ibrahim

Summer 2020

Contents

1 Introduction to Sequences

We will begin by introducing the notion of a *sequence of numbers*.

Definition (Sequence)

A *sequence* of numbers is an infinite list of numbers. We usually write the first few numbers of the sequence, and from that the pattern should be obvious enough (most of the time.) For example, the sequence of even numbers can be written as:

$$0, 2, 4, 6, \ldots$$

A number in the list, or sequence, is usually referred to as a *term* of the sequence.

It is important to know that it's not always the case that someone can guess the pattern with the first few terms. For example, let's consider the representation

$$1, 3, 9, \ldots$$

One might look at this and say that the sequence is the powers of 3, however they may be mistaken. These three terms also happen to be the first three terms of the sequence $T_N = 2N^2 + 1$. Observe:

N	0	1	2	3	4	$\cdots$
3^N	1	3	9	27	81	$\cdots$
$2N^2 + 1$	1	3	9	19	33	$\cdots$

As you've probably already guessed from the formulas above, we can represent sequences in ways other than listing the first few terms. Above we defined an *explicit* formula for each sequence. More on those later.

First, some notes on notation. When we refer to the terms of a sequence, we label them with an upper case letter and a subscript. Common uppercase letters include T (Term), P (Population), or A (Arithmetic). The subscript is the position in the list, starting with the 0th position.

Example (Notation) Let's look at the following sequence:

$$4, 7, 10, 13, \ldots$$

We will use T to denote a term. So we have the *first term* as $T_0 = 4$, the *second term* as $T_1 = 7$, the *third term* as $T_2 = 10$, and the *fourth term* as $T_3 = 13$. Notice how the first term is written with a *zero* subscript, and in general the Nth term is written as T_{N-1} with the subscript being one less. So the 100th term is written as T_{99}. This is because we start from 0.

2 Arithmetic Sequences and Linear Growth

2.1 Introduction and Formulas

Let's look at our sequence from the previous Example (Notation): $4, 7, 10, 13, \ldots$. This sequence is called an *arithmetic*, or *linear* sequence.

Definition (Arithmetic Sequence)

An *arithmetic* sequence, also referred to as a *linear* sequence, is a sequence where every pair of consecutive terms has a *common difference*. In other words, the **same number** (the common difference) is added to each term to get to the next term in the sequence.

We use the variable name d for the common difference. The easiest way to obtain d is by the formula

$$d = T_1 - T_0.$$

Notice this is just taking the difference between the second term T_1 and the first term T_0. If we know more terms, we could do $T_{50} - T_{49}$, as long as the terms are **consecutive**. So for the above sequence $4, 7, 10, 13, \ldots$ the common difference

$$d = T_1 - T_0 = 7 - 4 = 3.$$

Just to drive the point home,

$$d = T_2 - T_1 = 10 - 7 = 3,$$

and notice that adding 3 to each term does give the next.

As mentioned in the introduction, there are ways of representing a sequence without writing the first few terms. We will learn of two ways, both of which are useful at different times.

Definition (Recursive Formula)

A *recursive formula* will find a term of a sequence if we know the previous term. For example, if we want to know the 10th term of a sequence, in order to use the recursive formula we must know the 9th term.

The recursive formula for a linear sequence is

$$T_N = T_{N-1} + d.$$

In english, this formula says "The Nth term is the previous term (the $(N-1)$th term) plus the common difference." It's using what we know about the common difference, that is, we add d to a previous term to get to the next.

Now let's use the recursive formula on our sequence. We know that the fourth term, $T_3 = 13$, and our common difference $d = 3$. If we want to find T_4 using the recursive formula, we do

$$T_4 = T_3 + d = 13 + 3 = 16.$$

Now that we have T_4, we can find T_5 using the formula again:

$$T_5 = T_4 + d = 16 + 3 = 19.$$

The process can continue on forever.

What if we wanted to know the term T_{50} (the 51st term.) We would have to use the recursive formula 45 times! This is where the second formula comes in.

Definition (Explicit Formula)

An *explicit formula* for a sequence will find a term of the sequence using only information about its position (subscript).

For a linear sequence, the explicit formula is

$$T_N = T_0 + Nd.$$

In english: "Any term is equal to the first term plus the position of the term times the common difference." Notice how this looks like the formula for a line $y = mx + b$, hence the name *linear*.

$$\overbrace{T_N}^{y} = \overbrace{T_0}^{b} + \overbrace{N}^{x}\overbrace{d}^{m}$$

Let's see how to use this formula is useful to us.

Example (Applying the Explicit Formula, Arithmetic) Going back to our sequence, if we want to find T_{50}, we use the explicit formula. We know $T_0 = 4$ and $d = 3$, so plugging in we have

$$T_{50} = T_0 + Nd = 4 + (50)(3) = 154.$$

Using the explicit formula we find that the 51st term, denoted T_{50}, is 154. Suppose we want the 54th term. Then we do

$$T_{53} = T_0 + Nd = 4 + (53)(3) = 163.$$

One last thing. What happens if we are given a number and need to figure out which term it is in a sequence?.

Question. The number 48 is what term of the sequence $4, 7, 10, 13, \ldots$?

Solution. Another way to read this question is "for which N is $T_N = 49$?" We use the explicit formula and set up an equation. Let's just write it down, then analyze it:

$$\overbrace{49}^{T_N} = \overbrace{4}^{T_0} + \overbrace{N}^{\substack{\text{Need}\\\text{to}\\\text{find}}} \cdot \overbrace{3}^{d}.$$

We want to know which term, so particularly the value N, that 49 is of the sequence. If we solve the equation

$$\begin{aligned} 49 &= 4 + N \cdot 3 \\ 45 &= N \cdot 3 && \text{(Subtract 4)} \\ 15 &= N && \text{(Divide by 3)} \end{aligned}$$

So 49 is the 16th term, or $T_{15} = 49$ in the sequence. Let's check our work:

$$T_{15} = 4 + (15)(3) = 4 + 45 = 49.$$

Try it on your own. Find out which term the numbers 82 and 142 for the sequence $4, 7, 10, 13, \ldots$. The answers should be $N = 26$ and $N = 46$ respectively.

Summary

An arithmetic sequence is a sequence with a common difference d between any two consecutive terms.

- Recursive formula: $T_N = T_{N-1} + d$, used for quickly finding terms close to what we already know.
- Explicit formula: $T_N = T_0 + Nd$, used for quickly finding terms in much higher positions than we already know. Also, given a number, we can use the explicit formula to set up an equation and find out what term that number is in the sequence.

2.2 Linear Sums: Adding Terms Together

Sometimes we are interesting in finding the sum of the terms of a sequence. While we can compute the sums of every other term, or, for example, every fourth term, we will focus on adding up the first N terms of a sequence.

Sum Formula for Arithmetic Sequence

The formula for finding the sum of the first N terms of an arithmetic sequence is

$$\sum_{i=0}^{N-1} T_i = \overbrace{T_0 + T_1 + T_2 + \cdots + T_{N-1}}^{N \text{ terms}} \frac{(T_0 + T_{N-1})N}{2}.$$

The $\sum$ symbol is the capital version of the greek letter sigma, and it is used to denote a sum of terms. So we read the above formula as "the sum from i equals zero to $N-1$ of T_i." As an illustration, if we wanted the sum of the first **five** terms of a sequence, it would be written as

$$\sum_{i=0}^{4} T_i = \overbrace{T_0 + T_1 + T_2 + T_3 + T_4}^{5 \text{ terms}} = \frac{(T_0 + T_4)N}{2}.$$

The main take away is that the $\sum$ notation is used to signify what is being added, since if we're adding together 100 terms we don't want to write all that out across the page! The right hand side of the equation is what we can use to directly compute the sum, using the first added term, the last added term, and the number of added terms as the only information.

Finally, as a kind reminder, T_0 is the first term of the sequence, T_{N-1} is the Nth term of the sequence, and N is the number of terms being added together.

Now seeing that altogether is a bit confusing, so let's do an example.

Example (Sum of Arithmetic Sequence Terms) Consider our favorite sequence $4, 7, 10, 13, \ldots$, or as we've found explicitly: $T_N = 4 + N \cdot 3$. Let's do something we can check easily.

Question. What is the sum of the first four terms?

Solution. Well, we can directly compute $4 + 7 + 10 + 13 = 34$, but let's test out our formula. We have that $T_0 = 4$, $T_{N-1} = 13$, and $N = 4$. So the equation looks like:

$$\sum_{i=0}^{3} T_i = T_0 + T_1 + T_2 + T_3 = \frac{(T_0 + T_{N-1})N}{2} = \frac{(4+13)\cdot 4}{2}.$$

Notice that we want to know the sum of the first **four** terms, and the sum goes from $i = 0$ to $N-1 = 3$ (so that's four terms, T_0, T_1, T_2, T_3.) Now we can compute the sum, obeying the order of operations:

$$\sum_{i=0}^{3} T_i = \frac{(4+13)\cdot 4}{2} = \frac{(17)\cdot 4}{2} = \frac{68}{2} = 34.$$

What do you know, the formula works.

It's always nice to do some simple examples to get our feet wet. Try working the sum of the first four terms of the arithmetic sequence $1, 5, 9, 13\ldots$. The answer should be 28.

Now let's do a complete example, which will be slightly different.

Example (Arithmetic Sequence from Scratch) Consider the sequence $92, 85, 78, 71, \ldots$. Find the sum of the first fifteen (15) terms.

Solution. We want the sum of the first 15 terms of the above sequence. In sigma notation this is written as

$$\sum_{i=0}^{14} A_i$$

(here we are using A for "arithmetic.") According to our formula we have

$$\sum_{i=0}^{14} A_i = \frac{(A_0 + A_{14})N}{2}.$$

Well, we know $A_0 = 92$, and the number of terms we are adding is $N = 15$. All that remains is finding A_{14}. Since we only have 4 terms of our sequence, and we want the 15th term (A_{14}), let's find the **explicit formula** and use that (remember: the recursive formula would require us to have A_{13} first to obtain A_{14}.)

In the previous section we found that the explicit formula for an arithmetic sequence is

$$T_N = T_0 + Nd.$$

We know $N = 15$ and $A_0 = 92$, so we have

$$A_{14} = 92 + (15)d.$$

All that remains is to find the common difference d, which we know as

$$d = A_1 - A_0 = 85 - 92 = -7.$$

Notice that d is **negative**, as we would expect for a sequence that is decreasing (this is important, d is always a term minus the *previous* term.)
Now we have all the ingredients. Finally,

$$A_{14} = 92 + (15)(-7) = 92 - 98 = -6.$$

We are now ready to compute the sum. We have $N = 15, A_0 = 92, A_{14} = -6$, and so

$$\begin{aligned}\sum_{i=1}^{14} A_i &= \frac{(A_0 + A_{N-1})N}{2} \\ &= \frac{(A_0 + A_{14})(15)}{2} \\ &= \frac{((92) + (-6))(15)}{2} \\ &= \frac{(86)(15)}{2} \\ &= \frac{1290}{2} \\ &= 645.\end{aligned}$$

Whenever we want to add up the first N terms of some sequence, we can directly compute the sum once we find all the necessary ingredients like above. Try working the sum of the first 17 terms of the sequence $4, 7, 10, 13, \ldots$. The answer should be 476.

Summary

The formula for finding the sum of the first N terms of an **arithmetic** sequence is

$$\sum_{i=0}^{N-1} T_i = \overbrace{\frac{(T_0 + T_{N-1})N}{2}}^{\text{Main Part}}.$$

Sometimes we will only be given some terms of the sequence, and will have to work our way to the sum by first finding the common difference d and the last term in the sum.

2.3 Linear Growth Applications: Word Problems

Now that we've got arithmetic sequences in our toolbox, let's apply our skills.

Example (Movie Theater Rows) A movie theater has 32 rows of seats. There are 16 seats in the first row, 20 seats in the second row, and 24 seats in the third row. Assume this pattern continues.

1. How many seats are in the 10th row?
2. How many seats are in the 32nd row?
3. How many total seats are there in the theater?

Solution. Let's examine the information we have. We know the number of seats in the first three rows, so let's list them in order starting from the first row: $16, 20, 24, \ldots$. Ah-ha, there is a sequence! Let's call our terms "R" for "row", where $R_N =$ the number of seats in row N. So $R_0 = 16, R_1 = 20$ and $R_2 = 24$. (Remember, the "third" row is R_2, always 1 less in the subscript!)

So we must find the pattern to answer the questions. Assuming the pattern continues as the question says, notice that the number of seats in each row goes up by 4 from the previous row. This is our common difference, d. Indeed:

$$d = 20 - 16 = 4,$$

(or if you'd prefer $d = 24 - 20 = 4$.) Since we only have 3 terms and we want the 10th and 32nd terms, it is best to find the **explicit** formula.

We know that $d = 4$ and $R_0 = 16$, and the **explicit formula** for an arithmetic sequence is

$$R_N = R_0 + Nd,$$

(remember, it's explicit because the *only* thing we plug in is N when we find the specific formula.) So here we have

$$R_N = 16 + N \cdot 4.$$

Now we can find the number of seats in the 10th and 32nd row:

$$R_9 = 16 + (9)(4) = 52 \qquad \text{(10th row)}$$
$$R_{31} = 16 + (31)(4) = 140 \qquad \text{(32nd row)}$$

Well that takes care of 1. and 2., what about 3.? We are asked to find **the total** number of seats in the theater. This is different than finding just one term!

The problem states there are 32 rows total in the theater. So we want the sum of the first 32 terms of our general sequence. From our sum formula we have

$$\text{Total seats} = \sum_{i=0}^{31} R_i = \frac{(R_0 + R_{31})N}{2}.$$

This is “the sum of the number of seats of all 32 rows in the theater.” Notice it goes from 0 to 31. We have R_{31} from 2. (note problems will not always be this convenient), R_0, and N. For the actual calculation:

$$\frac{(R_0 + R_{31})N}{2} = \frac{(16 + 140)(32)}{2} = \frac{(156)(32)}{2} = 2496 \text{ seats.}$$

So in total we have:

1. How many seats are in the 10th row? – $R_9 = 52$
2. How many seats are in the 32nd row? – $R_{31} = 140$
3. How many total seats are there in the theater? – $\sum_{i=0}^{31} R_i = 2496$ seats.

That wasn't so bad. We had a scenario and came up with a sequence to solve some questions. Let's check out something a bit more complicated.

Example (College Enrollment) Beginning in the year 2009, scientists have been tracking the amount of penguin poop in Antarctica. For 2009, there were 325 tons of poop, for 2010 there were 350 tons of poop, and for 2011 there were 375 tons of poop. Assume this pattern continues.

1. How many tons of penguin poop were produced in the years 2012, 2016, and 2019?
2. Predict how many tons of penguin poop will be produced in 2030.
3. What is the total amount of penguin poop produced from 2009 to 2018 (“to 2018” means including that produced in 2018)?

For 1., we do exactly as before. First our sequence is $325, 350, 375, \ldots$. Let P_N be the tons of poop for year N, where the year 2009 corresponds to $N = 0$ (so 2009 is the first year, then 2010 is $N = 1$, and so on.) Any year's subscript after 2009 will be that year minus 2009. For example, 2015 is P_6, since $2015 - 2009 = 6$, So $P_0 = 325, P_1 = 350$, and $P_2 = 375$. As you've probably already guessed, the common difference $d = 25$, since $P_1 - P_0 = 350 - 325 = 25$.

For the year 2012 (P_3), we should use the **recursive formula**, since we already know the answer for 2011 (P_2). The recursive formula tells us

$$P_N = P_{N-1} + d.$$

So we have

$$P_3 = P_2 + 25 = 375 + 25 = 400.$$

Easy enough? If we want to get to the next term, we just add the common difference to the previous term, and here *we knew* what that previous term was.

We are not so lucky with the years 2016 (P_7) and 2019 (P_9). Let's find the explicit formula, given by $P_N = P_0 + Nd$. We know $P_0 = 325$ and $d = 25$, so now we just substitute for each N value.

$$P_7 = 325 + (7)(25) = 500 \qquad \text{(Year 2016)}$$
$$P_{10} = 325 + (10)(25) = 575 \qquad \text{(Year 2019)}$$

To predict how many tons of penguin poop will be produced in 2030, we use the explicit formula again, where 2030 is the term P_{21} (21 years after 2009, or $2030 - 2009 = 21$. So there will be tons of poop in 2030.

$$P_{21} = 325 + (21)(25) = 850 \qquad \text{(Year 2030)}$$

Finally, we answer 3. The total amount of penguin poop produced from 2009 to 2018 is the sum of the **first 10** terms of the sequence, starting with P_0 (2009) and ending with P_9 (2018). From our sum formula what we want to find is

$$\sum_{i=0}^{9} P_i = \frac{(P_0 + P_9)N}{2}.$$

We know P_0 and N, but we must find P_9. Using the explicit formula like above we have

$$P_9 = P_0 + N(25) = 325 + (9)(25) = 550.$$

So in total, we have

$$\sum_{i=0}^{9} P_i = \frac{(325 + 550)10}{2} = \frac{(875)(10)}{2} = 4375 \text{ tons.}$$

That's a lot of poop.
So we have

1. How many tons of penguin poop were produced in the years 2012, 2016, and 2019? – $400, 475$, and 575 tons respectively.

2. Predict how many tons of penguin poop will be produced in 2030. – 850 tons

3. What is the total amount of penguin poop produced from 2009 to 2018 ("to 2018" means including that produced in 2018)? – 4375 tons.

3 Geometric Sequences and Exponential Growth

3.1 Introduction and Formulas

Previously we discussed *arithmetic*, or linear sequences and how they can be used to represent "linear" growth. They main reason we say "linear" is because the "rule" to arithmetic sequences say we add a common difference d to one term to get to the next.

Now let's consider the sequence $5, 10, 20, 40, \ldots$. If we attempt to find a **common** difference, we will fail. Notice, $T_1 - T_0 = 10 - 5 = 5$, but $T_2 - T_1 = 20 - 10 = 10$. Since there is no common difference, this is not an arithmetic sequence.

We will now explore a new type of sequence, called a *geometric* sequence.

Definition (Geometric Sequence)

A *geometric* sequence, also referred to as an *exponential* sequence, is a sequence with a common ratio r between any two consecutive terms. In order to get the next term of a sequence, we **multiply** the previous term by r to get to the next term. The formula for the common ratio is

$$r = \frac{T_1}{T_0},$$

but just like the common difference, the ratio can be found using any two consecutive terms, e.g. $r = \frac{T_5}{T_4}$.

Back to our sequence $5, 10, 20, 40, \ldots$. Without knowing about geometric sequences, you would probably say the "rule" for this sequence is to double the previous term to get to the next. Indeed, the common ratio

$$r = \frac{\overbrace{10}^{T_1}}{\underbrace{5}_{T_0}} = \frac{\overbrace{20}^{T_2}}{\underbrace{10}_{T_1}} = \frac{\overbrace{40}^{T_3}}{\underbrace{20}_{T_2}} = \cdots = 2$$

Definition (Recursive Formula)

The *recursive* formula for a geometric sequence is

$$\overbrace{T_N}^{\substack{\text{Next}\\ \text{Term}}} = \overbrace{T_{N-1}}^{\substack{\text{Previous}\\ \text{term}}}(r).$$

Notice this is the rule that we stated before: "to get to the next term, you must multiply the previous term by the common ratio r."

Let's see how we apply the recursive formula.

Example (Applying the Recursive Formula) In our sequence $5, 10, 20, 40, \ldots$ we have four terms, T_0, T_1, T_2, and T_3. What if we wanted the fifth term, T_4? Using the recursive formula, we have

$$T_4 = T_3(r) = (40)(2) = 80.$$

You can think of the recursive formula as "what would I do to compute the terms one by one in my head."

Like we've seen, the recursive formula doesn't cut it on it's own. For higher order terms (order is just a fancy word for the position, or subscript), we can rely on the explicit formula.

Definition (Explicit Formula)

The *explicit* formula of a geometric sequence is

$$\overbrace{T_N}^{\substack{\textbf{Any}\\ \text{term}}} = \overbrace{T_0}^{\substack{\text{First}\\ \text{term}}}(r^N).$$

Again, like the explicit formula for arithmetic sequences, the only information we need is the subscript of the term we're looking for, the first term T_0, and the common ratio r.
In English: "**Any** Nth term is equal to the first term times r to the Nth power."
In other words, "times r, N times over."

Why does the explicit formula work, or make sense? In a **general** geometric sequence suppose we know the common ratio r. We start off with the first term T_0. Using the recursive formula, we can get T_1 by multiplying T_0 by r, i.e. $T_1 = T_0(r)$. Now to get T_2, we repeat the process; multiply T_1 by r, i.e. $T_2 = T_1 r$. But now notice this:

$$\begin{aligned} T_1 &= T_0(r) \\ T_2 &= T_1(r) = (T_0 r)(r) = T_0(r^2) \\ T_3 &= T_2(r) = (T_0 r^2)(r) = T_0(r^3) \\ T_4 &= T_3(r) = (T_0 r^3)(r) = T_0(r^4) \\ &\vdots \end{aligned}$$

Do you see the pattern? Start with $T_1 = T_0 r$, then since we know $T_2 = T_1 r$ by the recursive formula, and we know the relationship between T_1 and T_0, we can plug in for T_1 and get a relation between T_2 and T_0. This is the inner workings of the explicit formula, and why we use the word "exponential".

Just like the common difference d in the explicit formula for arithmetic sequences, the common ratio r is the same for all consecutive terms (hence the word "common".) All we really need to do to get **any** term is "start from the first term T_0 and multiply by r as many times as we need (N times for T_N)."

Enough with background, let's do an example.

Example Consider the sequence $5, 10, 20, 40, \ldots$. Let's find T_9. Using the explicit formula

$$T_9 = T_0(r^9) = (5)(2^9) = (5)(512) = 2560.$$

Let's also find T_{12}. Using the explicit formula,

$$T_{12} = T_0(r^{12}) = (5)(2^{12}) = (5)(4096) = 20480.$$

Now what if we were given a term and asked where in the sequence it occurs? (Remember, we did this for arithmetic sequences.) **This will not be expected of you, so you can skip over it if you want.**

Question. The number 5120 is what term of the sequence $5, 10, 20, 40, \ldots$?

Solution. Again, this question is asking "for which N is $T_N = 5120$. We set up an equation using the explicit formula:

$$5120 = 5(2^N)$$

and then we solve

$$\begin{aligned} 5120 &= 5(2^N) && \\ 1024 &= 2^N && \text{(Divide by 5)} \\ \log_2(1024) &= \log_2(2^N) && (\log_2 \text{ of both sides}) \\ \log_2(1024) &= N \log_2(2) && \text{(log properties)} \\ 10 &= N && \text{(Calculator)} \end{aligned}$$

So 5120 is the 11th term, or T_{10} of the sequence.

In an arithmetic sequence, we have two kinds of "progressions." If the common difference is *positive*, our sequence is *increasing*. If the common difference is *negative*, our sequence is *decreasing*. What about geometric sequences and the common ratio?

- If $r > 1$, then a geometric sequence is **increasing**.
- if $0 < r < 1$ (this is strictly bigger than 0 and strictly less than 1), then a geometric sequence is **decreasing**.

This is **essential** to understanding geometric sequences and exponential growth. Let's explore why through some examples.

Example (Common Ratios Effect) Consider the sequence $1, 3, 9, 27, \ldots$. The common ratio is

$$r = \frac{3}{1} = 3.$$

As we can see, $r > 1$, and the sequence is increasing. Most would agree that this is pretty straightforward. Let's get r a little closer to 1. What about the sequence $3, 3.3, 3.63, 3.993, \ldots$? The common difference is (you can use a calculator)

$$r = \frac{3.3}{3} = 1.1,$$

and we are much closer to 1, but still increasing. Let's use the recursive formula to understand this. For $r = 1.1$ we have $T_N = T_{N-1}(1.1)$. Another way to see this is to write it as

$$T_N = (1)T_{N-1} + (0.1)T_{N-1},$$

remember that combining like terms gets us back to $(1.1)T_{N-1}$. So the common ratio $r = 1.1$ is telling us that the next term is the previous term plus 0.1 times the previous term. In other words, the next term is an **increase of 10%** over the previous term.

$$\overbrace{T_N}^{\text{Next term}} = \overbrace{T_{N-1}}^{\text{Previous term}}(1.1) = \overbrace{(1)T_{N-1}}^{\text{Previous term}} + \overbrace{(0.1)T_{N-1}}^{\text{10\% of Previous term}}.$$

Now let's see what $r < 1$ looks like.

Consider the sequence $20000, 5000, 1250, 312.5, \ldots$. The common ratio is

$$r = \frac{5000}{20000} = \frac{1}{4} = 0.25,$$

here the fraction or the decimal are acceptable. Notice that our common ratio is less than 1 and the sequence is decreasing. Why? Since the common ratio is 0.25, or $\frac{1}{4}$, the next term in the sequence is only a **fraction** of the previous term:

$$T_N = T_{N-1}(0.25) = \frac{T_{N-1}}{4},$$

and since we only take a portion of the previous term at each step, we decrease.

Now let's get a little closer to 1. Consider the sequence $4, 3.8, 3.61, 3.4295, \ldots$. The common ratio is

$$r = \frac{3.8}{4} = 0.95,$$

much closer to 1 this time. This is telling us that the next term is 95% of the previous term, or 5% less than the previous term.

$$\overbrace{T_N}^{\text{Next term}} = \overbrace{T_{N-1}}^{\text{Previous term}}(0.95) = \overbrace{(1)T_{N-1}}^{\text{Previous term}} - \overbrace{(0.05)T_{N-1}}^{\text{5\% of Previous term}}.$$

Notice this time we split 0.95 into 1 and 0.05, and used subtraction. This is where the "5% less" comes from. So if the next term is always 5% less than the previous, we will decrease.

Notice that we didn't talk about $r = 1$ or $r \leq 0$. For $r = 1$ the sequence would just stay the same (multiplying by 1), for $r = 0$ all our terms after T_0 would be 0. We will not focus on $r < 0$.

As a quick side note, you will encounter phrases in word problems such as "increase of 5%," or "decreases by 10% yearly." When using geometric sequences to solve problems and we have a change of p percent (where p is in decimal form):

- The common ratio $r = 1 + p$ if it is an increase.
- The common ratio $r = 1 - p$ if it is a decrease.

For example, an increase of 5% would give a common ratio of 1.05, and a decrease of 20% would give a common ratio of 0.80.

Summary

A geometric sequence is a sequence with a common ratio r between any two consecutive terms. We have formulas for finding their terms.

- Recursive formula: $T_N = T_{N-1}r$, used for quickly finding terms close to what we already know.
- Explicit formula: $T_N = T_0(r^N)$, used for quickly finding terms in much higher positions than we already know.

The sequence increases or decreases based on the value of r.

- If $r > 1$, then a geometric sequence is **increasing**.
- if $0 < r < 1$ (this is strictly bigger than 0 and strictly less than 1), then a geometric sequence is **decreasing**.

3.2 Exponential Sums: Adding Terms Together

Like arithmetic sequences, we are interested in finding the sum of the terms of a sequence. We will focus on adding the first N terms of a sequence.

Sum Formula for Geometric Sequence

The formula for finding the sum of the first N terms of a geometric sequence is

$$\sum_{i=0}^{N-1} T_i = \frac{T_0(r^N - 1)}{r - 1}$$

Remember, the left side is just notation for "what is being computed,"; we are taking the sum from $i = 0$ to $N - 1$ of the terms T_i. In other words, we are summing the first N terms,

$$\sum_{i=0}^{N-1} T_i = \overbrace{T_0 + T_1 + T_2 + \cdots + T_{N-1}}^{N \text{ terms}}.$$

The right side gives us a small, convenient computation involving the first term T_0, the common difference r, and the total number of terms we are adding N.

Now that we have a sum formula for geometric sequences, let's take it for a test drive. Remember to be careful with order of operations, exponents, and paying attention to $N - 1$ on top of the sigma and the exponent N in r^N.

Example (Sum of Geometric Sequence Terms) Consider the sequence $5, 10, 20, 40, \ldots$. We found that the explicit formula is $T_N = 5(2^N)$. Let's do something we can check easily.
Question. What is the sum of the first four terms?

Solution. Directly adding them together we have $5 + 10 + 20 + 40 = 75$. Using the sum formula:

$$\sum_{i=0}^{3} T_i = \frac{5(2^4 - 1)}{2 - 1} = \frac{5(16 - 1)}{1} = 5(15) = 75$$

That was a nice little example. Try finding the sum of the first ten terms of the sequence in the above example. The answer should be 5115. Let's do a complete example.

Example (Geometric Sequence from Scratch) Consider the sequence $20000, 5000, 1250, 312.5, \ldots$. Find the sum of the first fifteen (15) terms.

Solution. Earlier we found the common difference to be $r = \frac{5000}{20000} = \frac{1}{4} = 0.25$. Now we setup our formula

$$\sum_{i=0}^{14} T_i = \frac{T_0(r^{15} - 1)}{r - 1}.$$

Remember, we are adding the first fifteen terms ($N = 15$), so the top of the sum should be $N - 1 = 14$. Unlike arithmetic sequences, there's no "later term" that we have to find. With just T_0 and r, we're good to go. So (we can use a calculator):

$$\sum_{i=0}^{14} T_i = \frac{(20000)((0.25)^{15} - 1)}{0.225 - 1} = \frac{(20000)(9.3132 \times 10^{-10} - 1)}{-0.75} = 26666.6666\ldots.$$

What a spooky number. It's important to pay attention to the order of operations when using a calculator.

Try working on finding the sum of the first 8 terms of the sequence $1, 4, 16, 64, \ldots$. The answer should be 21845.

Summary

The formula for finding the sum of the first N terms of a **geometric** sequence is

$$\sum_{i=0}^{N-1} T_i = \frac{T_0(r^N - 1)}{r - 1}.$$

This is more convenient since all we need is the common ratio and N, the number of terms we want to add up. The less convenient part is the actual computation, which may require a calculator for wacky numbers.

3.3 Exponential Growth Applications: Word Problems

Now that we've got geometric sequences in our toolbox, let's apply our skills.

Example (Marcia's Salary) Marcia has just been offered a job with a starting salary of \$32,000. She is not initially impressed, but the hiring committee is trying to persuade her by explaining that if she takes the job she is guaranteed a 7.5% raise each year for the next 20 years as long as her job performance is satisfactory. Assume that Marcia takes the job and starts in January 2020.

1. What will Marcia's Salary be in 2021, 2022, and 2023?
2. What will Marcia's Salary be in 2035?
3. How much **total** money will Marcia have been paid after working for the company for 20 years?

Solution. We can solve these problems by first defining a sequence. Let's use M_N for Marcia's Salary N years after 2020. So, $M_0 = 32000$, Marcia's salary in the year 2020 (0 years after 2020).
If we wanted to know how much she made in 2021, or M_1, we would use the **recursive** formula. The problem states that she is guaranteed a **7.5% increase** every year. This means that if x is her salary for some year, then the next year she will receive

$$x + \overbrace{0.075}^{7.5\%} x = 1.075x.$$

So the common ratio is $r = 1.075$ (remember your skill reviews, if a number x is increased by 7.5% then the new number is $1.075x$.) Now we have our recursive formula,

$$M_N = M_{N-1}(1.075).$$

Now we can calculate her salaries for 2021, 2022, and 2023.

$$\begin{aligned}
M_0 &= 32000 \\
M_1 &= M_0(1.075) = (32000)(1.075) = 34400 && (2021) \\
M_2 &= M_1(1.075) = (34400)(1.075) = 36980 && (2022) \\
M_3 &= M_2(1.075) = (36980)(1.075) = 39753.50 && (2023)
\end{aligned}$$

Things aren't so simple for 2035, however. Now we should use the **explicit** formula. The explicit formula is

$$M_N = M_0(r^N) = (32000)(1.075^N),$$

so for the year 2035, or M_{15}, we have

$$M_{15} = (32000)(1.075^{15}) \approx 94684.08.$$

Finally, we want to know the total money Marcia has been paid after working at the company for 20 years. The key word here is total. Since M_N is Marcias yearly salary N years after 2020, we want to add up 20 terms of our sequence to know how much she has been paid in total after 20 years of working. The setup is:

$$\sum_{i=0}^{19} M_i = \frac{M_0(r^{20}-1)}{r-1} = \frac{(32000)((1.075)^{20}-1)}{(1.075)-1}$$

Remember, N is the number of terms we are adding together, and here $N = 20$, or 20 years worth of salary. So the sum must go from 0 to $N - 1 = 19$. Compute the sum using a calculator. The answer should be $\$1,385,749.80$
So in total we have

1. What will Marcia's Salary be in 2021, 2022, and 2023? – \$34,400, \$36,980, \$39,753.50
2. What will Marcia's Salary be in 2035? – \$94,684
3. How much **total** money will Marcia have been paid after working for the company for 20 years? – \$1,385,749.80