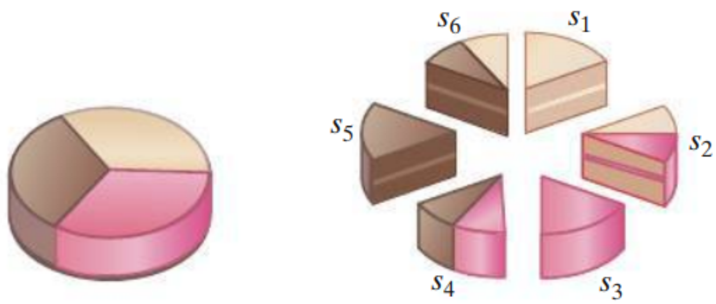
Example

Here we go through Exercise 13 on Pg. 92 of the ninth edition of the textbook.

13. Karla and five other friends jointly buy the chocolate-strawberry-vanilla cake shown below for \$30. After much discussion, the cake is divided into six shares, $s_1, s_2, \ldots, s_6$.

    Suppose Karla values strawberry cake *twice* as much as vanilla cake and chocolate cake *three* times as much as vanilla cake.



(a) (b)

(a) Find the dollar value to Karla of each of the slices s_1 through s_6.

Solution

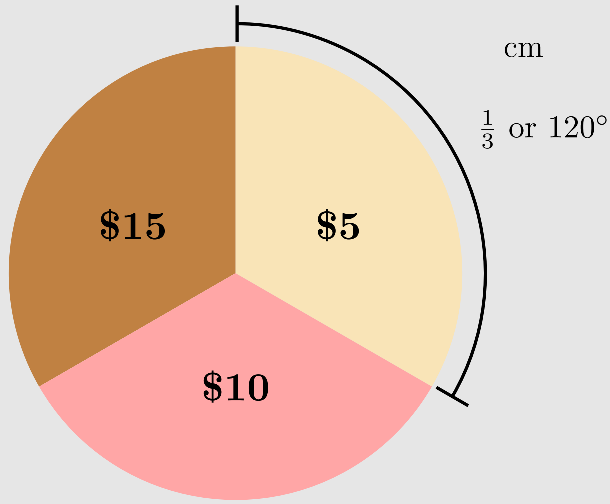
First we need to find how much each flavor portion is worth to Karla. Let s be the price of strawberry (in Karla's eyes), c be chocolate, and v be vanilla. Since Karla values strawberry *twice* as much as vanilla and chocolate *three times* as much as vanilla we have:

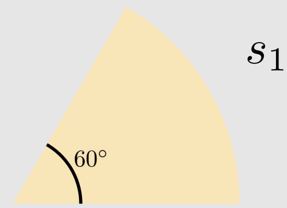
$$s = 2v$$
$$c = 3v.$$

If we add up the prices of each portion we should get the price of the total cake, in this case \$30. So we have

$$\begin{aligned} s + c + v &= \$30 \\ (2v) + (3v) + v &= \$30 \qquad \text{(Substitute in)} \\ 6v &= \$30 \\ v &= \$5 \end{aligned}$$

and since $v = \$5$, we get that $s = 2(\$5) = \$10$ and $c = 3(\$5) = \$15$. Checking our work, $\$5 + \$10 + \$15 = \$30$. Now to figure out the price of each share. Notice that each flavor portion is $\frac{1}{3}$ or $\frac{120°}{360°}$ of the cake.





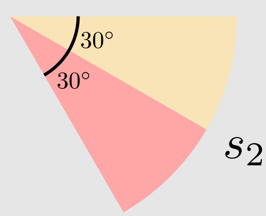
Each share is of equal **size**, and since we have 6 shares, or slices, each slice is $\frac{1}{6}$ of the original cake. So now we need to figure out what fraction of the vanilla s_1 is. Since s_1 is $\frac{1}{6}$ of the original cake, and the vanilla portion is $\frac{1}{3}$ of the original cake, it must mean that s_1 is $\frac{1}{2}$ the vanilla portion. Why? Observe:

$$\overbrace{\frac{1}{3}}^{\text{Vanilla portion}} \times \frac{1}{2} = \overbrace{\frac{1}{6}}^{s_1}$$

In other words, the above equation is saying "half of the vanilla portion of the cake is a sixth of the entire cake." This may all seem slightly complicated, so here is a simpler way to think about it. The vanilla portion is 120° of the entire cake. Since s_1 is $\frac{1}{6}$ of the entire cake, it is also 60° of the entire cake. So s_1 must be

$$\frac{s_1}{\text{vanilla}} = \frac{60^\circ}{120^\circ} = \frac{1}{2}$$

of the vanilla portion. Since $v = \$5$, the price of s_1 is $\frac{\$5}{2}$ or \$2.50.
What about a slice made up of two flavors? Let's look at s_2.



Remember, each share is 60° of the original cake. Since s_2 is exactly half vanilla half strawberry, each flavor portion of the slice must be 30° a piece. In other words we have 30° of vanilla and 30° of strawberry. So now we do exactly what we did for s_1, just twice and add. Observe:

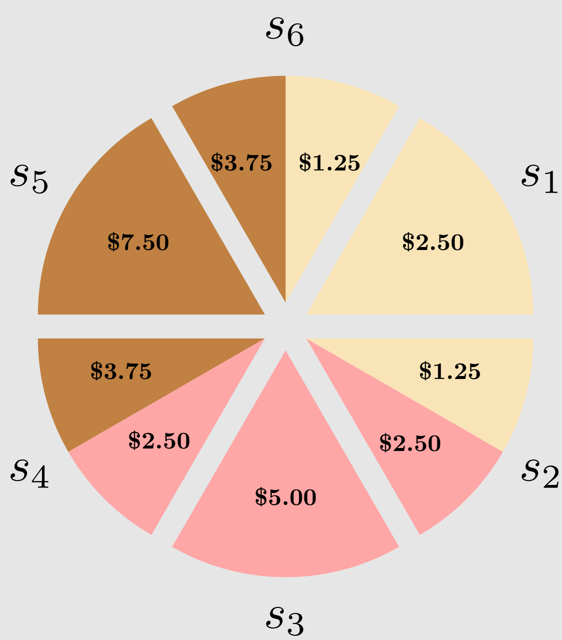
$$\frac{s_2 \text{ vanilla}}{\text{cake vanilla}} \times (\text{vanilla price}) = \frac{30^\circ}{120^\circ} \times (\$5) = \frac{1}{4}(\$5) = \$1.25$$

$$\frac{s_2 \text{ straw}}{\text{cake straw}} \times (\text{straw price}) = \frac{30^\circ}{120^\circ} \times (\$10) = \frac{1}{4}(\$10) = \$2.50$$

So the total cost of s_2 is $\$1.25 + \$2.50 = \$3.75$.
If we wanted to stick to fractions instead of degrees, remember we found that a share (which is a sixth of the original cake) is half of a flavor portion. Since s_2 is split equally into two flavors, we have

$$\overbrace{\frac{1}{2}}^{\frac{s_2}{\text{flavor}}} \times \overbrace{\frac{1}{2}}^{s_2 \text{ is half-half}} = \frac{1}{4}$$

strawberry, and the same for vanilla. Here is a picture summarizing the results:



Don't forget to quickly check your work! Adding all the prices in the above picture should give you \$30.

$$
\begin{aligned}
s_1 &= \$2.50 \\
s_2 &= \$1.25 + \$2.50 = \$3.75 \\
s_3 &= \$5.00 \\
s_4 &= \$2.50 + \$3.75 = \$6.25 \\
s_5 &= \$7.50 \\
s_6 &= \$\$3.75 + \$1.25 = \$5.00
\end{aligned}
$$

(b) Which of the slices s_1 through s_6 are fair shares to Karla?

Solution

Karla considers a slice to be a fair share if it is valued at $\frac{\$30}{6} = \$5$ in her eyes (total value divided by number of players.) So Karla considers s_3, s_4, s_5, and s_6 to be fair shares.