Chapter 10 – Financial Mathematics
Notes on Forms of Interest

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10 Financial Mathematics

10.1 Percentages

Most if not all of us are already familiar with the notion of a "percent." It is an intuition we build to understand things that are part of a whole.

Definition (Percentage)

A "fraction" with denominator 100 can be interpreted as a *percentage*, and the percentage symbol '%' is used to indicate the presence of the hidden denominator 100.

$$x\% = \frac{x}{100}$$

Percentages are especially useful when we want to compare several measurements that pertain to parts of a "different whole", or total. Let's look at an example.

Example You've taken a summer class in which there were three exams. Your scores are as follows:

Exam 1	22 out of 25
Exam 2	40 out of 50
Exam 3	100 out of 160

Each score above is called a *raw score*. Since each exam has a different point total, it is difficult to say which exam you performed best in. With percentages, however, we can compare results.

Exam 1	22 out of 25	$\frac{22}{25} = \frac{88}{100} = 88\%$
Exam 2	40 out of 50	$\frac{40}{50} = \frac{80}{100} = 80\%$
Exam 3	100 out of 160	$\frac{100}{160} = 0.625 = 62.5\%$

Using percentages, we can see that you performed best on Exam 1.

For the first two exams above, it was convenient for us to convert the fraction into one whose denominator is 100 by multiplying the numerators and denominators by 4 (denominator 25) and 2 (denominator 50) respectively. As for the third exam, we can resort to computing the fraction in decimal form (say, on a calculator) and then converting that to a percentage. This brings us to conversions:

- We can express a ratio $\frac{x}{y}$ as a percent by converting it into a decimal (do the division) and then multiplying by 100 (move the decimal two places to the right.)

- We can express $P\%$ as a decimal by dividing P by 100. This is regardless of whether or not P itself is a decimal (for example, $0.05\% = 0.0005$.) This can also be seen as "move the decimal point two places to the left."

Example We can convert the following:

1. $44\% = 0.44$
2. $0.775 = 77.5\%$
3. $253\% = 2.53$
4. $\frac{13}{14} \approx 0.9286 = 92.86\%$
5. $5\% = 0.05$
6. $0.34\% = 0.0034$

Now that we have an idea of what percentages are, let's take it a step further. How do we find $x\%$ of some quantity. For example, how does one find 15% of 200? The easiest way is to convert 15% to decimal and multiply it by 200. That is, 15% of 200 is

$$(0.15)(200) = 30.$$

This is a convenient rule. If we want to know how much $x\%$ of y is, then we convert $x\%$ to decimal form and multiply it by y.

This now allows us to obtain the skills we're going to need in the future. Consider the following example.

Example Sandy makes \$45,000 a year at her current job. She was just offered a higher position that will net her a **15% increase** in her yearly salary. What is the salary of the higher position?

Answer. The example states that the new position's salary is a 15% increase over her old salary. In order to compute the new salary, we take the old salary and add 15% of it to itself. If the old salary is s then we have

$$\begin{aligned} \text{New Salary} &= s + (0.15)s \\ &= \$45,000 + (0.15)(\$45,000) \\ &= \$45,000 + \$6,750 \\ &= \$51,750. \end{aligned}$$

We can generalize this to come up with a convenient rule for "percent changes."

Rule (Percent Changes)

Let x be some number, and let r be the decimal form of $P\%$. Then:

- If we want to increase x by $P\%$ to get y then $y = x + (r)x = (1 + r)x$.
- If we want to decrease x by $P\%$ to get y then $y = x - (r)x = (1 - r)x$.

We used a lot of variable names above, so let's use some concrete numbers to solidify our understanding. Let's say our "x" is 150 and $P\%$ is 20%. Then 20% in decimal form is 0.2, or r. So

- A 20% **increase** of 150 will give

$$150 + (0.2)(150) = (1 + 0.2)(150) = (1.2)(150) = 180$$

- A 20% **decrease** of 150 will give

$$150 - (0.2)(150) = (1 - 0.2)(150) = (0.8)(150) = 120$$

The main take away here is to convert the percentage to decimal form, add it or subtract it from 1 depending on what kind of change you are computing, and multiply by the original value.

Finally, be careful of falling into some traps when using percentages.

Example Let's see two examples.

1. Let B be your baseline salary at your job. Suppose one year you get a 5% raise and the next year you get a 20% raise. The two raises **do not** combine to be a 25% increase over B. Instead, we first increase B by 5% to get a new salary after the first raise, then we increase that new salary by 20% to get to the ultimate salary after two years. In terms of B the raises combined amount to

$$\overbrace{(1.20)\overbrace{(1.05)B}^{\text{New Salary}}}^{\text{Ultimate Salary}} = (1.26)B$$

or a 26% increase.

2. If the value of your house increases by 25% one year then decreases by 20% another year, the value of your house stays the same. If B is the baseline value for your house we have similar to the first example

$$\overbrace{(0.80)\overbrace{(1.25)B}^{\text{New Value}}}^{\text{Ultimate Value}} = (1)B = B$$

If the multiplication in the above examples does not make sense, the reader is encouraged to write out the steps, starting with the baseline value, then computing the first change, then computing the second change using the new value.

Summary

The main take away here is percent change. If we *increase* or *decrease* by $x\%$ we convert to decimal form (say r) and multiply by $1 + r$ or $1 - r$ respectively.

Interest Terminology

Similar to renting an apartment or car, there is a concept of renting currency itself. *Interest* is the price paid for the use of other's money. Before we move on to talking about different types of interest, we need to get some terminology under our belt.

(Definition (Terminology)

- **Principal** – Denoted P. This is the sum of money lent, or *initially* invested.
- **Interest rate** – Denoted r. This is a rate charged to use the principal P for some amount of time, usually measured in years. When the rate is expressed as a percentage and the unit of time is a year, the rate is called an **annual percentage rate (APR).**
- **Term** – Denoted t. The *term* of a loan, or length of time the money is being borrowed (i.e. the amount of time allotted to pay back the loan.) Typically measured in years, but can be months, weeks, or even days.
- **Repayment Schedule** – This is an agreed upon schedule for repayment of a loan. There are several types
  - In *single payment* loans the loan is repaid in one lump sum.
  - In *installment loans* the borrower repays the loan by making equal monthly payments over the term of the loan.
  - In *credit card loans* the repayment schedule is (up to a point) at the discretion of the borrower.

10.2 Simple Interest

The first type of interest we will explore is *simple interest*.

Definition (Simple Interest)

When a loan is based on simple interest, the interest rate is applied only to the principal P per unit of time (for example, once a year.) This means the borrower pays the same amount of interest each time period.

Sometimes it helps to translate a definition into a formula that we can more easily understand. In a simple interest based loan, the interest rate is applied only to the principal. Let r be the APR in decimal form and P be the principal. Then every time the borrower is charged interest, they are charged exactly

$$(r)(P)$$

dollars. If t is the life of the loan in **years** (remember, APR means *annual* percentage rate), then over the life of the loan the total interest is

$$\overbrace{I}^{\text{Interest}} = \overbrace{t}^{\substack{\text{Number}\\\text{of}\\\text{times}}} \overbrace{r}^{\substack{\text{Interest}\\\text{rate}}} \overbrace{P}^{\text{Principal}}$$

This is called the **simple interest formula.**

Definition (Simple Interest Formula)

The simple interest formula is given by

$$I = Prt$$

where I is interest, P is the principal, r is the APR in decimal form, and t is the life of the loan in years.

This is most of the work, except we now need a way of calculating a *final value* of a loan, which is given by

$$F = P + I$$

where F stands for final value. This should be clear; the amount we pay back at the end of a loan is the amount we borrowed, P, along with the interest, I. Using the simple interest formula to substitute for I, we can write this in two ways:

$$F = P + Prt$$

or

$$F = (1 + rt)P.$$

Either way works fine.

Example Suppose you receive an offer from a credit union in the mail. They offer you the chance to take out a loan under simple interest with an APR of 4%. You decide to take out a \$2000 loan with a term of 3 years. How much will you have to pay back?
Answer. The basic strategy for problems involving interest is to always list the givens. We are given an APR of 4%, so $r = 0.04$. The money lent is \$2000, so $P = \$2000$. Finally, the term is 3 years, so $t = 3$. The problem is asking for the amount of money we have to pay back, i.e. the *final value*. We've already seen that the final value is exactly what you'd expect: the principal plus the interest charged. Using the simple interest formula, the interest charged for this loan over 3 years is

$$I = Prt = (\$2000)(0.04)(3) = \$240.$$

Then adding that to the principal we have

$$F = P + I = \$2000 + \$240 = \$2240,$$

the amount we owe.

Some examples may not be as straightforward as the previous. For example, instead of a final value, we may be asked to find a principal or interest rate.

Example Suppose a credit union is offering loans under simple interest with an APR of 3.5%. How much should you borrow if wish to pay exactly \$100 in interest over 2 years?
Answer: We are given an APR of 3.5%, so $r = 0.035$, an interest of \$100, so $I = \$100$, and a term of 2 years, so $t = 2$. The problem is asking us how much we should initially borrow, in other words, we need to find the value of P. We have I, r, and t, so we should use the simple interest formula.

$$\begin{aligned} I &= Prt \\ \$100 &= P(0.035)(2) \\ \$100 &= P(0.035)(2) \\ \$100 &= P(0.07) \\ \$1428.57 &= P && \text{(Divide by 0.07)} \end{aligned}$$

In the above example we found a principal. Let's do one more example where we find a rate.

Example Suppose you take out a \$2000, 4 year loan under simple interest. At the end of the loan you pay back \$2160 in total. What is the APR?
Answer. Here we are given $P = \$2000$, $t = 4$, and the final value $F = \$2160$. In order to find the APR, we need to solve for r. We can do this in one of two ways. First, we can find the interest:

$$I = F - P = \$2160 - \$2000 = \$160.$$

Then we can use the simple interest formula:

$$\begin{aligned} I &= Prt \\ \$160 &= (\$2000)(4)r \\ \$160 &= (\$8000)r \\ 0.02 &= r && \text{(Divide by \$8000)} \end{aligned}$$

We can also use the formula for the final value:

$$\begin{aligned} F &= P + Prt \\ \$2160 &= \$2000 + (\$2000)(4)r \\ \$160 &= (\$2000)(4)r && \text{(Subtract \$2000)} \end{aligned}$$

and then we're eventually at an identical step in the first method. Although we found $r = 0.02$, we are specifically asked to find the APR, or annual *percentage* rate. Converting r to a percent gives an APR of 2%.

Of many loans available, there are specific types that operate under simple interest. One type is known as a *bond*.

Definition (Bond)

Bonds are a type of investment usually issued by a government, corporation, or municipality. When you purchase a bond, you lend your money to the agency issuing the bond. Like the loans in our previous examples, Bonds are usually defined by the term, APR and "face value" (the amount the agency pays back to you when the bond reaches *maturity*.)

Let's look at an example.

Suppose you purchase a ten-year bond with a face value of \$500 and APR of 2.5%. What is the original price of the bond?
Answer. Let's once again write down what is given. We have $t = 10$, $r = 0.025$,

and $F = \$500$. Remember, the *face value* of a bond is just another name for the final value of a loan under simple interest. We need to find the original purchase price of the bond, i.e. P. Setting up our formula:

$$\begin{aligned} F &= (1 + rt)P \\ \$500 &= (1 + (0.025)(10))P \\ \$500 &= (1.25)P \\ \$400 &= P && \text{(Divide by 1.25)} \end{aligned}$$

Notice the small return on the bond. We would make \$10 a year with those terms in the example. This is generally the case with bonds and other "safe" investments. Now we will take a look at a type of loan that many deem to be predatory.

Definition (Payday Loan)

A **payday loan** is a short-term loan, usually with a term of days or weeks, where one can borrow money using collateral. Forms of collateral include a future paycheck, tax return, personal check, etc.

Payday loans, also known as cash advances, are very prevalent in poorer communities, where people are typically in need of short term cash. Some make the association of poverty with lower financial literacy, hence why payday loans may be considered predatory (in addition to what we're about to see.)

Example Suppose it's the summer and your electricity bill has skyrocketed. You need \$300 or the electricity in your home will be shut off. You make your way to the neighborhood CashNow payday loan service. Using your future paycheck as collateral, you take out a 14-day, \$300 loan. CashNow charges \$20 interest for every \$100 borrowed. What is the APR of the loan?

Answer. Here we are given $P = \$300$ and $I = \$20 \times 3 = \$60$. In order to use our simple interest formula to find r, we need to know what t is. A quick mistake one would make is assign $t = 14$, but this is incorrect. Remember, we are looking for the APR, or *annual* percentage rate. So, t must be in years. We can view 14 days in years by dividing by 365. That is, $t = \frac{14}{365}$. Now we can proceed.

$$\begin{aligned} I &= Prt \\ \$60 &= (\$300)\left(\frac{14}{365}\right)r \end{aligned}$$

and solving for r gives

$$r = \frac{\$60}{(\$300)\left(\frac{14}{365}\right)} = 5.214$$

So our APR is r in percentage form, or 521.4%.

Notice how large the APR is. It is typical of payday loans to have such a large APR, and is another reason why they are considered predatory. However, we would be remiss if we neglect to mention that there is tremendous risk for the lender involves as well. For example, a future paycheck given as collateral may not clear.

One last note to make is on the *growth* of a loan under simple interest. Let's investigate the final value formula in the form $F = P + Prt$. Let's say the principal and interest rate are fixed, and t can be changed. In simple interest, the interest rate is only applied to the original principle, i.e. Pr is constant. So we have

$$\overbrace{F}^{T_N} = \overbrace{P}^{T_0} + \overbrace{Pr}^{d}\overbrace{t}^{N},$$

or the explicit formula for linear growth. In other words, the final value of a loan under simple interest grows linearly with time, with a common difference $d = Pr$. You can see this if you come up with an example with a fixed principal and APR, and you play with t.

Summary

For simple interest we have "two" formulas, the main simple interest formula

$$I = Prt$$

where I is the interest, P is the principal, r is the APR, and t is the term. Using the simple interest formula we can obtain a final value formula:

$$F = P + I = P + Prt = (1 + rt)P$$

where F is the final value.

10.3 Compound Interest

The next type of interest we will look at is compound interest. Unlike simple interest, where the interest rate is applied only to the principle, compound interest uses a new principle every time interest is compounded.

Definition (Compound Interest)

A loan under compound interest is a loan where the interest rate is applied to a new principle each time interest is compounded. That is, we start with an original principle P, apply the interest rate to get a final value F, then the next time F becomes our new principle.

We will begin by discussing annual interest in an example.

Example Suppose that on the day you were born your parents deposit \$4000 into a college fund account that pays 8% APR, and the interest is compounded annually at the end of each year.

1. On the day you were born, there is \$4000 in the account. This is the principal.

2. On your first birthday, the amount in the account grows. The new balance is: (1.08)(\$4000).

3. On your second birthday, we use the current balance to calculate interest. That is, we earn 4% of (1.08)(\$4000), not the original \$4000. This gives a new balance of $(1.08)(1.08)(\$4000) = (1.08)^2(\$4000)$

The pattern continues each year. By now you can probably guess that after N years, there will be $(1.08)^N(\$4000)$ in the account.

Note that, if you are say 18 years and 10 months old, then the account balance would be $(1.08^{18})(\$4000) = \$15,984.08$, and there would be no "partial interest" for the 10 months. That is, if you wait 2 more months the interest would compound once more.

What would happen if we double the principal? What would happen if we double the APR? These are important questions for understanding the power of compound interest. Taking our numbers from the earlier example:

$$(1.08^{18})(\$4000) = \$15,984.08 \qquad \text{(Original)}$$
$$(1.08^{18})(\$8000) = \$31,968.16 \qquad (2P)$$
$$(1.16^{18})(\$4000) = \$57,850.06 \qquad (2r)$$

Notice that doubling the principal exactly doubled the final value in the end. This is expected, since

$$(1.08^{18})(2 \times \$4000) = 2 \times \overbrace{(1.08^{18})(\$4000)}^{\text{Original}}.$$

Closer examination on doubling the APR reveals that our final value has more than tripled. This is generally the case with compound interest, and hints at the fact that we are dealing with *exponential growth*. Try the doubling the APR and using $t = 50$.

Definintion (Annual Compound Interest Formula)

From the previous example we've seen that we can generalize to obtain the formula

$$F = P(1 + r)^t$$

for the final value of a loan under annual compound interest. Note that t has to be a whole number of years, where we round *down* if it's not since we only compound at the *end* of a year.

Now let's look at a type of investment that utilizes compound interest.

Definition (Certificate of Deposi)

A **certificate of deposit**, or CD, is a type of investment that is a hybrid between a savings account and a bond. A CD is a safe investment with a fixed term and fixed APR, usually compounded annually, but could also be semiannual or monthly.

The above definition is the first mention of compounding other than annually. Indeed, interest can be compounded more than once a year. Let's first start with another annual example.

Example Suppose you invest \$2000 on a CD with an APR of 4.3% compounded annually. What is the future value of the CD after one year? After two years?
Answer: We have $P = \$2000$ and $r = 0.043$. After one year the future value on the CD

$$F = (\$2000)(1 + 0.043)^1 = (\$2000)(1.043) = \$2086.$$

After two years we have

$$F = (\$2000)(1 + 0.043)^2 = (\$2000)(1.043)^2 = \$2175.70$$

Now what if we wanted to compound interest more than once a year? Let's use the previous example.

Example Use the same principal of \$2000 and APR of 4.3%. Suppose the CD compounds quarterly. Find the future value after one and two years.
Answer. Compound quarterly means four times a year. However, our APR (*annual* percentage rate) is 4.3%. Compounding quarterly doesn't mean applying this 4.3% rate four times a year. Instead, we take the APR and divide it by the number of times we compound. In this case we have

$$\frac{4.3\%}{4} = 1.075\%.$$

Converting this percentage to decimal we have 0.01075. This value is known as the *periodic* interest rate, usually denoted with a p (not to be confused with principal, P.) Now we can compute the final value for each *quarter* (let's call these Q_i):

$$\begin{aligned} Q_1 &= (\$2000)(1.01075) = \$2021.50 \\ Q_2 &= (\$2021.50)(1.01075) = \$2043.23 \\ Q_3 &= (\$2043.23)(1.01075) = \$2065.20 \\ Q_4 &= (\$2065.20)(1.01075) = \$2087.40 \end{aligned}$$

So at the end of one year, or four quarters, our final value is \$2087.40. This is more than our previous final value of \$2086 under annual compounding. For the value after two years, let's use our heads. Looking at one year, we've multiplied by the periodic interest rate four times. In other words, the final value for one year was

$$F = (1.01075)^4(\$2000).$$

One could venture a guess and say for two years we'd have

$$F = (1.01075)^8(\$2000) = \$2178.61.$$

Again, this is more than we received under annual compounding for the same amount of time!

Under a fixed APR and term, the more often interest is compounded the better it is for the investor. However, comparing values between the first example and the one above, one may notice that the benefit is not significant. Of course, compounding quarterly is not the only variant, and we can give a general formula.

Definition (General Formula for Compound Interest)

The general formula for the final value of a loan under compound interest can be given in two ways:

$$F = P(1+p)^T$$

$$F = P\left(1+\frac{r}{n}\right)^{nt}$$

Where F is the final value, P is the principal, r is the interest rate, $p = \frac{r}{n}$ is the periodic interest rate, n is the number of times compounded in a year, t is the length of the loan in years, and $T = nt$, the number of times compounded over the life of the loan.

Notice that in the formula above, if we compound annually ($n = 1$) we obtain the annual compound interest formula discussed earlier. Here is a table showing the different values of n for different compounding frequencies.

Compound	n
Annual	1
Semiannual	2
Quarterly	4
Monthly	12
Daily	365

To finish off, let's do an example to examine each frequency.

Example Suppose you invest \$2500 on a CD with an APR of 2.5%. Find the future value of the CD after 3 years under the five previously mentioned compounding frequencies.
Answer. We can use the general compounding formula

$$P\left(1+\frac{r}{n}\right)^{nt}.$$

We have $P = \$2500$, $r = 0.025$, and $t = 3$. So

$$F = (\$2500)\left(1 + \frac{0.025}{1}\right)^{(1)(3)} = (\$2500)(1.025^3) = \$2692.23 \qquad \text{(Annual)}$$

$$F = (\$2500)\left(1 + \frac{0.025}{2}\right)^{(2)(3)} = (\$2500)(1.0125^6) = \$2693.46 \qquad \text{(Semiannual)}$$

$$F = (\$2500)\left(1 + \frac{0.025}{4}\right)^{(4)(3)} = (\$2500)(1.00625^{12}) = \$2694.08 \qquad \text{(Quarterly)}$$

$$F = (\$2500)\left(1 + \frac{0.025}{12}\right)^{(12)(3)} = (\$2500)(1.00208\overline{33}^{36}) = \$2694.50 \qquad \text{(Monthly)}$$

$$F = (\$2500)\left(1 + \frac{0.025}{365}\right)^{(365)(3)} = (\$2500)(1.0000684^{1095}) = \$2694.70 \qquad \text{(Daily)}$$

Notice how the final value increases the more we compound in a year, however the benefits are marginal.

Summary

The general formula for the final value of a loan under compound interest is

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

where F is the final value, P is the principal, r is the interest rate, n is the number of times compounded in a year, and t is the length of the loan in years.

10.5 Consumer Debt

Contemporary consumer debt consists of two common types of loans — credit cards and installment loans (think mortgages).

10.5.1 Credit Cards

Unlike previously discussed investments or loans, credit cards are a bit special. Other than a possible annual fee to use a card, interest is not charged unless there is a *balance* due at the end of a billing cycle (usually a month.) If the balance is not paid in full by the payment due date, then the remaining balance is called a *revolving balance* and finance charges will be incurred based on interest during the *next* billing cycle. We will see how the finance charge is computed in an example. Finally, late payments incur a late fee.

This is interesting. If we use a credit card to make purchases throughout a month, but pay our balance in full before the payment due date, then we owe no interest. This is a convenient way to borrow money.

Example Leorio decided to take an island vacation and use his credit card with a 24% APR for expenses. Things were a bit expensive throughout his vacation and he ended up charging more than he can pay off in full on his credit card. For the June 1 – June 30 billing cycle, Leorio is unable to pay the full balance and has a revolving balance of \$600. This balance is carried over to the next cycle, July 1 – July 31. Here are the charges for the July cycle:

Date	**Charge**	**Amount**
1 July	Previous Balance	\$600
12 July	Medical Supplies	\$50
15 July	Necktie	\$20
24 July	Cellphone Bill	\$45

Since the previous balance was not paid in full, we need to compute a *finance charge* to see the amount owed for the month of July. We apply simple interest to the *average daily balance.* In order to do so, we need to calculate the period interest rate and average daily balance.
For the periodic interest rate p we have the formula

$$p = \frac{\text{Days}}{365} \times r$$

where r is the APR in decimal form and "Days" is the number of days in the billing cycle. We have $r = 0.24$, and the number of days in our billing cycle is 31 (July 1 - July 31). So the periodic interest rate

$$p = \left(\frac{31}{365}\right)(0.24) \approx 0.02038.$$

Now for the average daily balance. It's exactly how it sounds. We need to find the average balance over all the days in the billing period.
Looking at the table of charges, a new charge was not made until July 12. This means that on **each day** including and between July 1 through July 11 the balance was \$600. Then on July 12, there was a \$50 charge, raising the balance to \$650. This would be the balance for July 12, July 13, and July 14. Then there is a new charge on July 15. This pattern continues, and we can make a table.

Period	Daily Balance	Number of Days
July 1 - July 11	\$600	11
July 12 - July 14	\$650	3
July 15 - July 23	\$670	9
July 24 - July 31	\$715	8

Now we're able to calculate the **average** daily balance. Remember, calculating an average means adding up all data points, then dividing by the number of data points

$$\frac{(11)(\$600) + (3)(\$650) + (9)(\$670) + (8)(\$715)}{31} \approx \$654.84.$$

If you need a visual of the daily balance for each day:

July 1: \$600	July 9: \$600	July 17: \$670	July 25: \$715
July 2: \$600	July 10: \$600	July 18: \$670	July 26: \$715
July 3: \$600	July 11: \$600	July 19: \$670	July 27: \$715
July 4: \$600	July 12: \$650	July 20: \$670	July 28: \$715
July 5: \$600	July 13: \$650	July 21: \$670	July 29: \$715
July 6: \$600	July 14: \$650	July 22: \$670	July 30: \$715
July 7: \$600	July 15: \$670	July 23: \$670	July 31: \$715
July 8: \$600	July 16: \$670	July 24: \$715	

Now to calculate the finance charge we multiply the average daily balance by the periodic interest rate:

$$\text{Finance Charge} = \$654.84 \times (0.02038) = \$13.35.$$

We can also calculate the new balance for the next billing cycle by summing up the previous balance, purchases, and the finance charge. The new balance in this case:

$$\text{New Balance} = \$600 + \$115 + \$13.35 = \$728.35$$

We need to make some important notes here. The number of days in a *billing cycle* is 30 or 31, depending on the month. A *period* (like in our daily balance table in the example) starts the day a charge is made (or the first day of the cycle) and ends the day **before** the next charge is made (or the last day in the cycle.) One must be careful while counting the number of days as well. For example, July 12 - July 14 is 3 days (12, 13, and 14), however $14 - 12 = 2$. To combat this, if the two dates you are subtracting have the same month a quick calculation involves subtracting the days and adding 1, e.g. $14-12+1=3$. The reader is encouraged to look at the table in the above example and note when the periods begin and end and how they correspond to the charges made on the card.

There are many benefits to using a credit card, however one must be careful of the pitfalls. One can avoid finance charges by paying their balance in full each month, however it is useful to pay attention to the APR, payment due dates, late fees, etc.

10.5.2 Installment Loans

Another type of debt consumers incur are known as installment loans. These loans allow consumers to make much bigger purchases than they can afford upfront, such as housing and car purchases. There is always risk in lending money to individuals, however with installment loans more often than not the items purchased would serve as collateral themselves (especially property, or houses.) This leads to lower APRs than things like credit cards and cash advances.

In an installment loan, the borrower pays the loan plus interest to the lender in equal installments paid at regular intervals. Think along the lines of monthly car or mortgage payments.

Definition (Amortization + Formula)

Amortization is the process of paying off a loan by making regular installment payments.
We will be looking at installment loans with monthly payments. The *amortization formula* is

$$M = P\left(\frac{p(1+p)^T}{(1+p)^T-1}\right),$$

where M is the monthly payment, P is the principal, $p=\frac{r}{12}$ is the monthly interest rate, and T is the life of the loan in months (the number of months interest is charged.)

Now that we have an idea of what installment loans are and how to compute a monthly payment, let's do an example.

Example Jim and John just bought a new house for \$355,000. They took out a 30-year mortgage with an APR of 3.1%. What is their monthly payment?
Answer. Let's begin by listing what we have. The principal $P = \$355,000$ and $r = 0.031$. In order to use the amortization formula, we need to compute p and T. For p we have

$$p = \frac{r}{12} = \frac{0.031}{12} \approx 0.0025833.$$

Next we have T, the length of the loan in months. Converting 30 years to months we have

$$T = (12)(30) = 360.$$

Plugging into the formula:

$$\begin{aligned} M &= P\left(\frac{p(1+p)^T}{(1+p)^T - 1}\right) \\ &= (\$355,000)\left(\frac{(0.0025833)(1+0.0025833)^{360}}{(1+0.0025833)^{360} - 1}\right) \\ &= (\$355,000)\left(\frac{(0.0025833)(1.0025833)^{360}}{(1.0025833)^{360} - 1}\right) \\ &= (\$355,000)\left(\frac{(0.0025833)(2.531441354)}{(2.531441354) - 1}\right) \\ &= (\$355,000)\left(\frac{0.00653947}{1.531441354}\right) \\ &= \$1,515.90 \end{aligned}$$

So the monthly payment $M = \$1,515.90$.

Let's analyze this for a moment. Jim and John purchases a home for \$355,000 by taking out a 30-year mortgage. Their monthly payment turns out to be \$1,515.90 as we calculated above. Over the life of the loan (360 months) they will have paid back

$$(\$1,515.90)(360) = \$545,724.$$

In other words, they've paid a staggering

$$\$545,724 - \$355,000 = \$190,724$$

in interest. That's \$6,357.47 in interest alone per year (divide the interest by 30.)

Let's do one more example for good measure.

Example Pam just bought a house. She made a \$15,000 **down payment** and financed the balance with a 20-year home mortgage loan with an interest rate of 5% compounded monthly. Her monthly mortgage payment is \$900. What was the selling price of the house?

Answer. Let's write out what we're given. We have $T = (12)(20) = 240$ months. The APR is 5%, so $r = 0.05$ and $p = \frac{r}{12} = \frac{0.05}{12} = 0.004166667$. The monthly payment $M = \$900$. So we need to find the selling price of the house. Let's ignore the down payment for now. We must find P. Setting up the equation:

$$
\begin{aligned}
M &= P\left(\frac{p(1+p)^T}{(1+p)^T-1}\right)\\
\$900 &= P\left(\frac{(0.004166667)(1+0.004166667)^{240}}{(1+0.004166667)^{240}-1}\right)\\
\$900 &= P\left(\frac{(0.004166667)(1.004166667)^{240}}{(1.004166667)^{240}-1}\right)\\
\$900 &= P\left(\frac{(0.004166667)(2.7126405)}{(2.7126405)-1}\right)\\
\$900 &= P\left(\frac{0.0113025217}{1.7126405}\right)\\
\$900 &= P(0.0065994712)\\
\frac{\$900}{0.0065994712} &= P\\
\$136,374.56 &= P
\end{aligned}
$$

We found the principal P, however we are not done. The amount \$136,374.56 was only the portion of the purchase price that was financed, since Pam put a down payment of \$15,000 (in other words an upfront payment to reduce the potential loan amount.) Adding the down payment to the principal gives us a purchase price of

$$\$136,374.56 + \$15,000 = \$151,374.56.$$

As we have seen from the examples, installment loans are long-term loans that can accumulate much interest over time, but also benefit from being paid early.

Summary

1. Credit Card Debt.

   - There is a *periodic interest rate* $p = \left(\frac{N}{365}\right) r$ where N is the number of days in a *billing cycle* and r is the APR in decimal form.
   - There is the average daily balance, computed by adding the daily balance for **each** day and dividing by the number of days in the billing cycle. We can make a table highlighting *periods* where the balance is constant to make things easier.
   - The *finance charge* is computed by multiplying the the average daily balance by the period interest rate.

2. Installment Loans.

   - Typically long-term loans for large purchases with low APRs.
   - The amortization formula is

$$M = P\left(\frac{p(1+p)^T}{(1+p)^T - 1}\right),$$

   where M is the monthly payment, P is the principal, $p = \frac{r}{12}$ is the monthly interest rate, and T is the life of the loan in months (the number of months interest is charged.)