Calculus I

Course Notes

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This work would not have been possible without the hard work and dedication of many individuals to make many technical things possible. This document was produced in LaTeX, making use of KOMA-script, as well as TikZ and PGF.

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π Preface

These course notes are typed up under the wire style. There are bound to be typos, or algebra errors. **Students who find such errors will be handsomely rewarded with extra credit.** The purpose of these notes are to provide students with a point of reference for concepts and examples particularly discussed in class, with potentially other bonus material.

Some of these notes are based on the time I taught Calculus I at Virginia Commonwealth University (known as Math 200 there), where we followed Calculus: Early Transcendentals (Third Edition, Briggs, Cochran, Gillett, Schulz). These notes will also be based on two sections of Calculus I that I am teaching at Lafayette College (Math 161), where we follow Calculus by Stewart, Clegg, and Watson (Ninth Edition).

0 Prerequisites

Students looking to brush up on their pre-reqs have many resources at their disposal.

- Paul's Online Math Notes – I have used these notes when learning calculus myself, and I find this repository to be an invaluable part of the internet.
- Precalculus Course Notes – I have written up and compiled some course notes from teaching precalculus at VCU. These notes would not have been possible without the help of many instructors and course coordinators at VCU. **Find errors? Extra credit.**
- Any Calculus textbook you can get your hands on. Usually these books have comprehensive pre-req sections and diagnostic tests with answers (in particular, the book for Math 161 contains all of those.)

1 What is a limit?

In this section we will discuss the idea of a limit. There are several ways to motivate this idea. Some of these ways are artistic in nature, while others have more tangible applications.

1.1 Motivations

1.1.1 Geometry - The Area of a Circle

Let's consider the following question.

Question 1.1

How can we calculate the area of a circle?

Certainly, one may remember that given a circle of radius r, we can compute its area A exactly with the formula

$$A = \pi r^2.$$

Let's assume we do not have that knowledge.

Definition 1.2: Polygons

A *polygon* is a closed figure in the plane consisting of edges and points where the edges meet (vertices). A *regular* polygon is a polygon with all angles equal and all sides of equal length.

Figure 1.1 has several examples of polygons.

Recall that the area of a triangle with a base length of b and a height of h is $(1/2)bh$. A perfectly wonderful and nice formula. However, there is another formula using the spooky sine function. If we have a triangle with two sides of length a and b, and the angle between those two sides is ϑ, then the area of the triangle is $(1/2)ab\sin(\vartheta)$. This

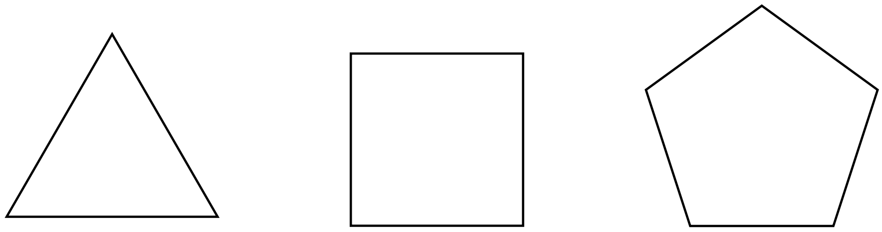


Figure 1.1: Three regular polygons, namely the triangle, the square, and the pentagon.

comes from the definition of sine and the area formula $(1/2)bh$. Review trig functions here.

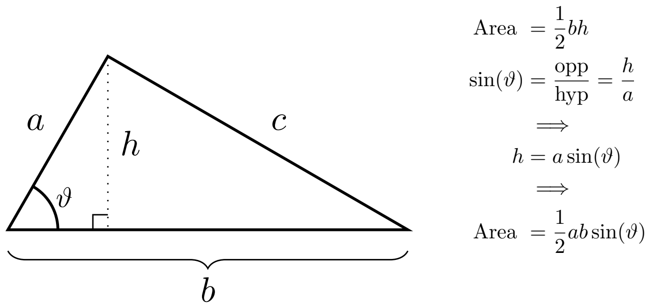


Figure 1.2: The derivation of the area of a triangle using the sine function.

With this knowledge, we can compute the area of any polygon, including irregular polygons, by cutting them up into triangles. For example with the pentagon in Figure 1.3. So taken any polygon in the plane, we can triangulate it and add up the area of

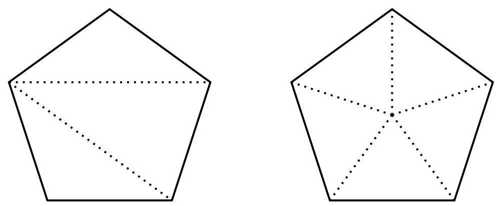


Figure 1.3: The pentagon triangulated in two ways.

each triangle in order to get the area of the polygon. In particular, we can triangulate any regular polygon by placing a point in the center of the polygon, and connecting that point to every vertex. In this way, with a bit of thinking, we can come up with a formula for the area of a regular pentagon with n sides and radius R, where the radius is the distance from the center point to any vertex of the polygon. Simply, we compute the area of each triangle, and we add up the total area. See Figure 1.4.

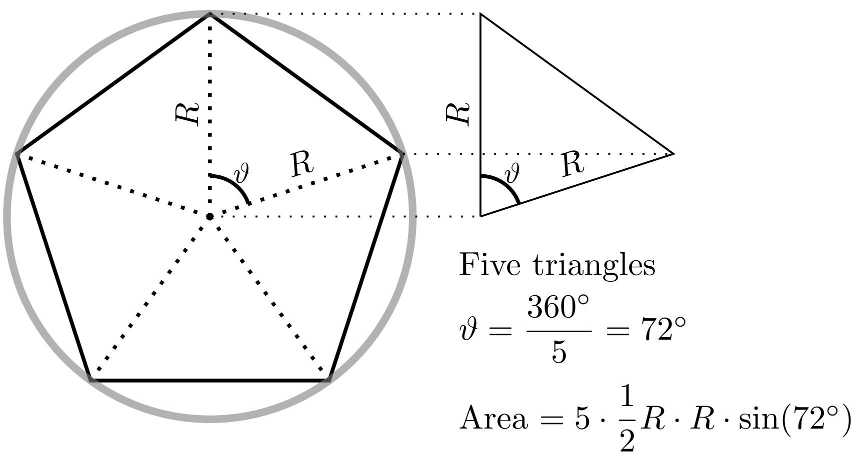


Figure 1.4: A regular pentagon partitioned into five equilateral triangles.

Using this knowledge, how does one compute the area of a circle? Well... with limits! As hinted by Figure 1.4, we can inscribe a regular polygon in a circle of interest. By computing the area of an inscribed polygon, we are able to approximate the area of the circle. Our general formula for the area A_n of a regular pentagon with n sides is

$$A_n = n \cdot \frac{1}{2} R^2 \sin\left(\frac{360^\circ}{n}\right).$$

Notice that the more sides our polygon has, the better our area approximation is. See Figure 1.5.

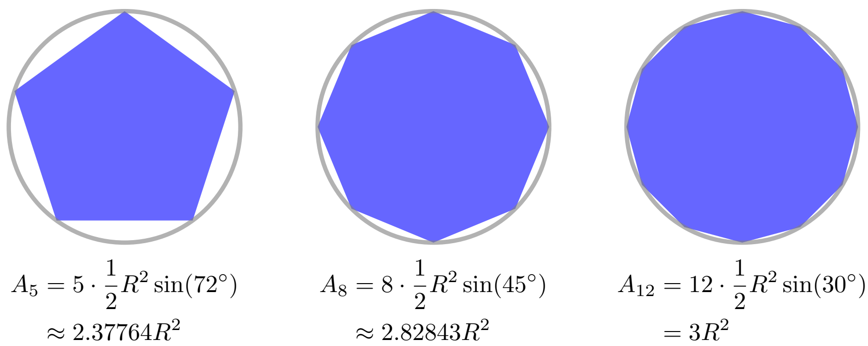


Figure 1.5: As the number of sides of the inscribed polygon increases, the closer one gets to the true area of the circle.

Notice that the coefficient of R^2, namely $n \frac{1}{2} \sin\left(\frac{360^\circ}{n}\right)$ is getting larger and larger with n. Computing this coefficient for larger values of n, we get:

n	A_n
5	$2.37764R^2$
8	$2.82843R^2$
12	$3R^2$
30	$3.11868R^2$
45	$3.13139R^2$
60	$3.13585R^2$
75	$3.13792R^2$
90	$3.13904R^2$
120	$3.14016R^2$
$\vdots$	$\vdots$
∞	πR^2

1.1.2 Curve Sketching and String Art

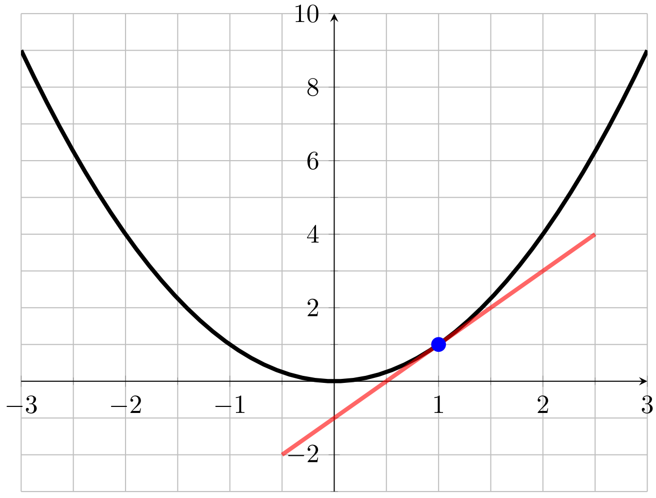
Section still under construction.

Let's begin with a hand-wavy definition of a tangent line.

Definition 1.3: Definition: Tangent Lines

The *tangent line* to a curve at a given point is the straight line that touches the curve at that point.

Below is an example of a line tangent to the curve $f(x) = x^2$ at the point $x = 1$.



A huge part of this course is understanding what a tangent line is, how to define tangent lines with calculus thinking, and what they relate to. So, given a curve, we can draw lines which are tangent to that curve. What about the other way around; can we draw tangent lines first and obtain a curve? This is where string art comes in!

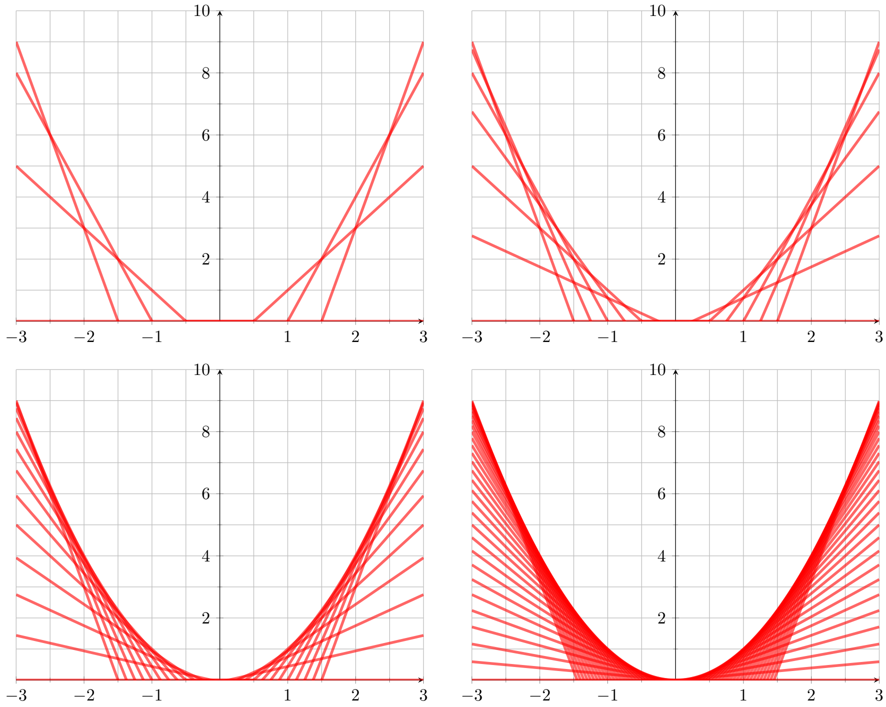


Figure 1.6: Hi I am string art

Notice that, the more strings we use, the closer we get to approximating some curve. You can mimic this exercise on any piece of paper. In the case of Figure 1.6, what kind of curve do you see popping up? To the naked eye, it may seem like we've drawn a smooth and perfect curve, but if we zoom in:

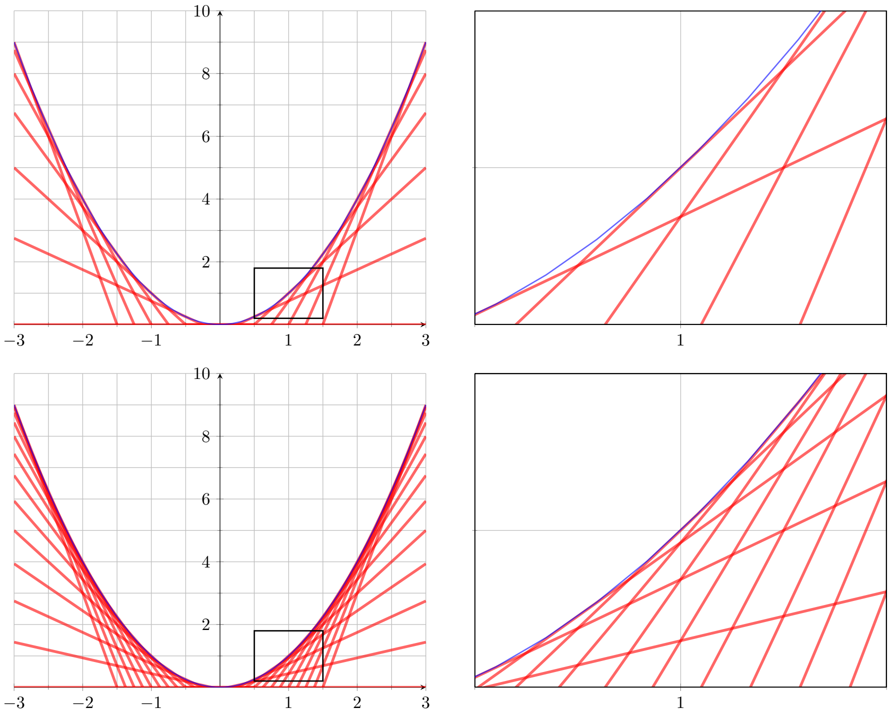


Figure 1.7: The more lines we use, the finer the curve we approximate.

As the number of lines increases, the closer we get to a smoother curve. After a certain point, the curve is indistinguishable from that of x^2 to our eyes. However, in the language of limits, as the number of lines we use approaches infinity, the curve that we sketch approaches that of x^2.

1.1.3 The Tangent Line and Instantaneous Velocity

Let us start this section with an exercise.

Problem:

Ray is driving from NYC to Richmond, Virginia. After one hour, Ray is 60 miles from NYC. After three hours, Ray is 160 miles from NYC. How fast is Ray driving?

After some thought, one might have several ways to answer the question. For example, one may say “Ray was 60 miles away from NYC after one hour of driving, therefore Ray traveled 60 miles in one hour, or at a speed of 60 miles per hour.” Another may say “After three hours, Ray was 160 miles away from NYC, therefore it took Ray three hours to travel 160 miles. Thus Ray is traveling at a speed of 160 miles$/3$ hours ≈ 53.33 miles per hour.” Finally, one might say “we can use the distance Ray traveled between hours one and three, which is $160 - 60 = 100$ miles, and divide that by two hours to get 50 miles per hour.”

These are all valid ways of thinking about this *disingenuous* question. However, it could have been possible that Ray drove a constant 70 miles per hour from the one hour mark to the two hour mark, and a constant 30 miles per hour (perhaps due to light traffic) from the two hour mark to the three hour mark. So what exactly is going on here? We are all aware that we very rarely drive at a constant speed; there are changes in traffic flow, traffic lights, changes in speed limits. Well, every computation we made so far has been an *average* speed.

$$\frac{60-0 \text{ miles}}{1-0 \text{ hours}} = 60\,\frac{\text{mi}}{\text{hr}} \qquad \frac{160-0 \text{ mi}}{3-0 \text{ hr}} \approx 53.33\,\frac{\text{mi}}{\text{hr}} \qquad \frac{160-60 \text{ mi}}{3-1 \text{ hr}} = 50\,\frac{\text{mi}}{\text{hr}}$$

Let’s look at a another scenario.

Scenario

A ball is launched upwards in the air from the ground. The height (in feet) of the ball is modeled as a function of time (seconds) by the function $s(t) = -16t^2 + 96t$.

Now, we can ask all sorts of questions. For example:

Question 1.4

What is the maximum height that the ball reaches in the air?

Well, the function s is a quadratic function whose graph is a parabola pointing downwards ( ). If we want to know what the maximum height of the ball is, then we need to compute the maximum value of this quadratic function. If we know anything about quadratic functions, it's that the maximum or minimum value occurs at the vertex. So we have

$$t = \frac{-b}{2a} = \frac{-96}{2(-16)} = 3,$$

which means that the ball reaches its maximum height after three seconds in the air. To find the maximum height itself, we plug this value in to the function, i.e.

$$s(3) = -16(3)^2 + 96(3) = 144 \text{ ft.}$$

Another question we may ask is:

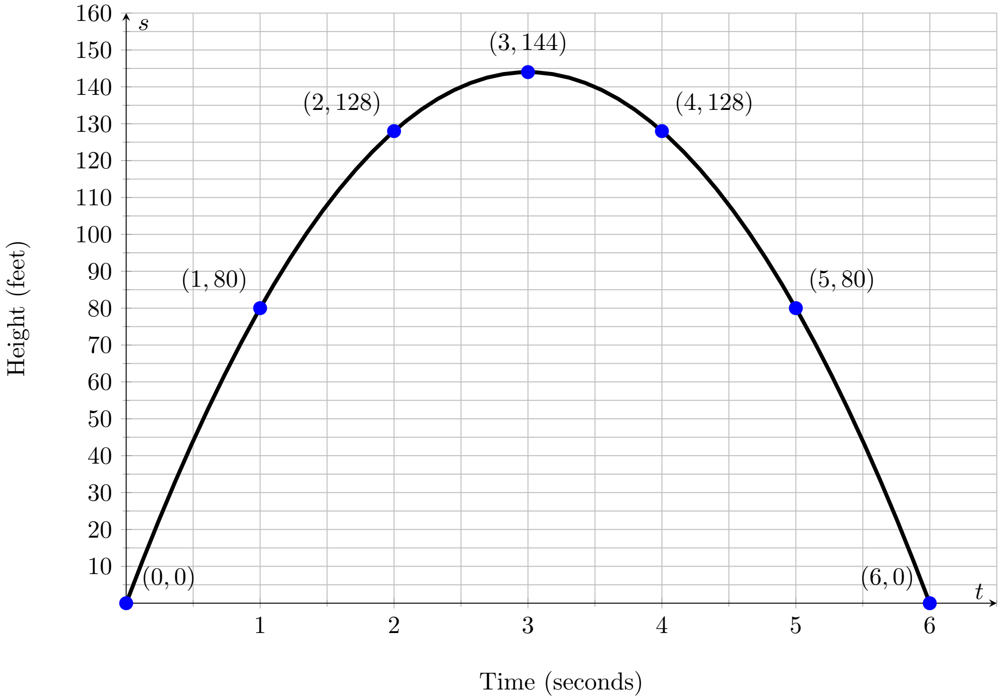
Question 1.5

At what point does the ball hit the ground, or equivalently, how long is the ball in the air?

In this case, since $s(t)$ is the height of the ball at time t, we can interpret the time that the ball hits the ground as a solution to the equation $s(t) = 0$. So, using our quadratic equation skills, we have

$$\begin{aligned} s(t) &= 0 \\ -16t^2 + 96t &= 0 \\ -16t(t-6) &= 0. \end{aligned}$$

Our two solutions are $t = 0$ and $t = 6$. Therefore, the ball is in the air for six seconds. With all our information, let's go ahead and plot a graph of our function.



Finally, let's talk about something a bit more interesting: the speed of the ball.

Question 1.6

How fast is the ball traveling from $t = 1$ seconds to $t = 2$ seconds?

First, a question we can answer with an intuitive understanding of physics.

Question 1.7

How fast is the ball traveling "at" $t = 3$ seconds?

As we've established, the ball reaches its maximum height at $t = 3$. In other words, the ball travels up starting from $t = 0$, to every unit of time right before $t = 3$, and then travels down from every time right after $t = 3$ to $t = 6$. However, in order to change direction from up to down, at some specific, or *instantaneous* point in time, the ball had to have stopped moving in the upwards direction. In this case, the ball stops at $t = 3$, and by stop we mean, the ball has a speed of 0 ft/s "at $t = 3$".

Now let's examine changes in the ball's velocity leading up to $t = 3$, the time it reaches peak height. The ball's average velocity from $t = 0$ to $t = 3$ is the difference in the ball's position, Δs, divided by the difference in the end and start time, Δt. In other words, the average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{s(3) - s(0)}{3 - 0} = \frac{144 - 0}{3} = 48 \text{ ft/s}.$$

We can do this for several different start times leading up to $t = 3$, and for times after. Let's make a table.

Time Interval	Average Velocity	Time Interval	Average Velocity
$[0, 3]$	$\frac{s(3) - s(0)}{3 - 0} = 48$ ft/s	$[3, 3.001]$	$\frac{s(3.001) - s(3)}{3.001 - 3} = -0.016$ ft/s
$[1, 3]$	$\frac{s(3) - s(1)}{3 - 1} = 32$ ft/s	$[3, 3.001]$	$\frac{s(3.01) - s(3)}{3.01 - 3} = -0.16$ ft/s
$[2, 3]$	$\frac{s(3) - s(2)}{3 - 2} = 16$ ft/s	$[3, 3.01]$	$\frac{s(3.1) - s(3)}{3.1 - 3} = -1.6$ ft/s
$[2.5, 3]$	$\frac{s(3) - s(2.5)}{3 - 2.5} = 8$ ft/s	$[3, 3.25]$	$\frac{s(3.5) - s(3)}{3.25 - 3} = -4$ ft/s
$[2.75, 3]$	$\frac{s(3) - s(2.75)}{3 - 2.75} = 4$ ft/s	$[3, 3.5]$	$\frac{s(3.75) - s(3)}{3.5 - 3} = -8$ ft/s
$[2.9, 3]$	$\frac{s(3) - s(2.9)}{3 - 2.9} = 1.6$ ft/s	$[3, 4]$	$\frac{s(4) - s(3)}{4 - 3} = -16$ ft/s
$[2.99, 3]$	$\frac{s(3) - s(2.99)}{3 - 2.99} = 0.16$ ft/s	$[3, 5]$	$\frac{s(5) - s(3)}{5 - 3} = -32$ ft/s
$[2.999, 3]$	$\frac{s(3) - s(2.999)}{3 - 2.999} = 0.016$ ft/s	$[3, 6]$	$\frac{s(6) - s(3)}{6 - 3} = -48$ ft/s

These computations seem awfully familiar…In each computation, we are computing a change in output, Δs, and dividing by a change in input, Δt. In other words, for two times t_0 and t_f we have two points $(t_0, s(t_0))$ and $(t_f, s(t_f))$, and we compute the quantity

$$\frac{s(t_f) - s(t_0)}{t_f - t_0},$$

which is the slope between those two points. So here's the punchline: relative to our curve $s(t) = -16t^2 + 96t$, we have been computing the slope of the secant line between two points; the average rate of change from one time to the next is the slope of the secant line connecting those two points. Furthermore, as we compute the average rate of change in a way where one endpoint of our interval is approaching $t = 3$, that is, the interval is getting shorter and shorter, notice that the slope of the secant line is approaching 0. That is, the slope of the secant line is approaching the speed of the ball "at" $t = 3$, or the *instantaneous* velocity of the ball.

Note On Speed vs. Velocity

We've been using the words speed and velocity interchangeably. The velocity is the speed of the ball, with the extra information of the direction. In one-dimensional motion, this information is presented as the sign $(\pm)$ of the velocity. For example, looking at the tables of our average velocity computations, we can see that when the ball is traveling up (away from the ground), the velocity is positive, whereas when the ball is traveling downward (toward the ground), the velocity is negative.

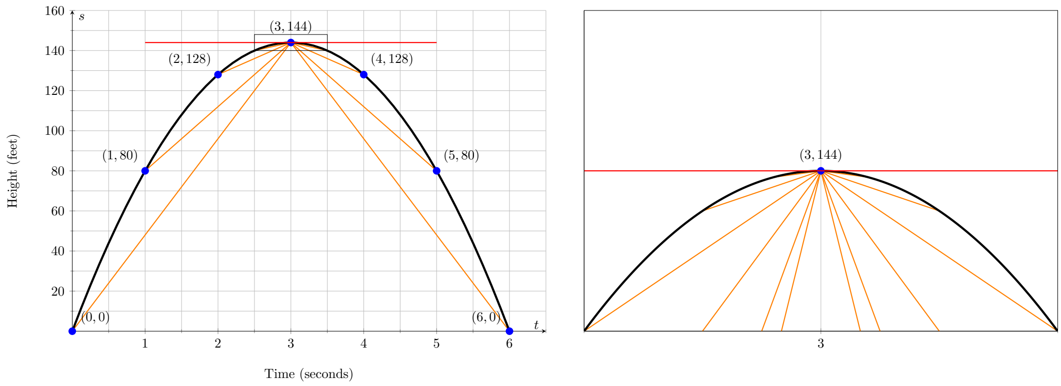


Figure 1.8: The sequence of secant lines approaching the line tangent to the curve at $t = 0$.

One can explore this function a bit more in desmos: Secant to Tangent.

We can now define the notion of *instantaneous* rate of change. The instantaneous rate of change of a function at a point is the slope of the line tangent to that point. What are tangent lines? Tangent lines are lines intersecting a curve at two infinitely close points on that curve. This presents us with the first revolutionary idea of calculus: the limit. We can write that the ball's instantaneous velocity at $t = 3$ is

$$\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = 0.$$

Definition 1.8: Tangent Line

The *tangent line* at a point $x = a$ of a curve is the line which intersects two points on the curve which are infinitely close to a. In limit notation, the slope of the tangent line $m_{\tan}$ at $x = a$ is given by

$$m_{\tan} = \lim_{x \to a} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This can be thought of as the evolution of the slopes of the secant lines, $m_{\sec}$ between $(a, f(a))$ and points $(x, f(x))$ as x gets closer to a from the left and the right.

In other words, instantaneous rate of change is the result of analyzing the evolution of average change over infinitely small intervals of input (in the case of the ball scenario, time.)

1.2 The Limit of a Function

What-to-review.

Before starting this section, it would be good to review: Functions (notation, domain, range, the graph of a function) and piecewise-defined functions. Also review the basics of trigonometric functions.

The limit of a function $f(x)$ as x approaches a value a is written

$$\lim_{x \to a} f(x).$$

This idea, or quantity, is the answer to the question "how does $f(x)$ behave for values of x which are infinitely close to a". If

$$\lim_{x \to a} f(x) = L$$

then we say "as x approaches a, the output $f(x)$ approaches L." The general idea of the limit of a function can be viewed with a table of values and graphs.

Example 1.9: Limit Through Table of Values

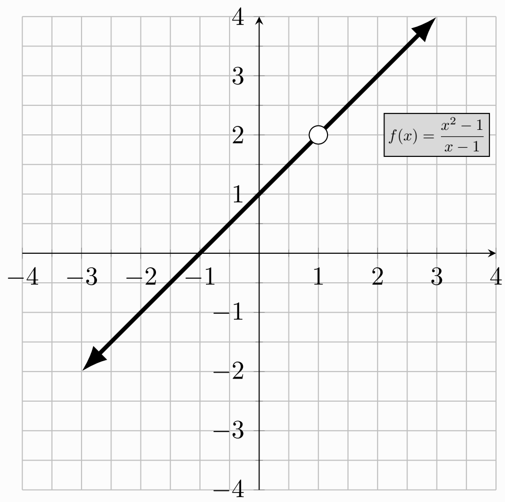
Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The keen-eyed student may look at this function and notice that we can simplify it a bit:

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x-1)(x+1)}{x-1} \\ &= x + 1. \qquad (x \neq 1) \end{aligned}$$

We have to be careful when simplifying; even though we were able to cancel factors from the numerator and denominator, the original definition of the function excludes 1 from the domain. So, this function defines a straight line with a hole at $x = 1$.



Now, $f(1)$ is undefined. However, we can still analyze the function around $x = 1$. In other words, as x approaches 1, how does the function $f(x)$ behave? Visually, this is not hard to see from the graph. As our x values approach 1 from the left, the value $f(x)$ gets closer and closer to 2. Similarly, as x values approach 1 from the right, the value $f(x)$ gets closer and closer to 2. We can see this using a table of values:

x	$f(x) = x + 1, x \neq 1$
0.500	1.500
0.750	1.750
0.900	1.900
0.990	1.990
0.999	1.999
1.000	Undefined
1.001	2.001
1.010	2.010
1.100	2.100
1.250	2.250
1.500	2.500

In this case, we say "as x approaches 1, the output $f(x)$ approaches 2." In limit notation, this is written

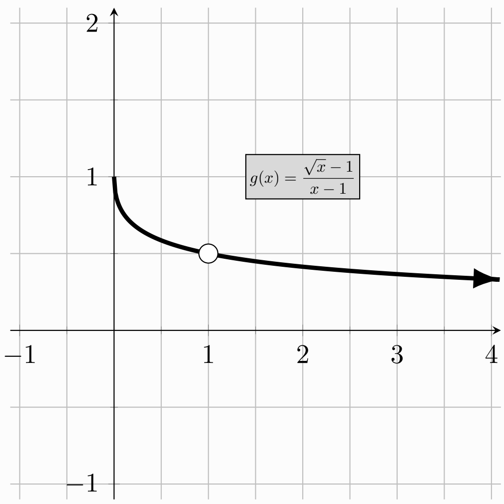
$$\lim_{x \to 1} f(x) = 2.$$

Example 1.10: Limit Through Table of Values 2

Consider the function

$$g(x) = \frac{\sqrt{x} - 1}{x - 1}.$$

Notice that $x = 1$ is not in the domain of g, as the denominator cannot be equal to 0.



x	$g(x) = (\sqrt{x}-1)/(x-1)$
0.500	0.5858
0.750	0.5359
0.900	0.5132
0.990	0.5013
0.999	0.5001
1.000	Undefined
1.001	0.4999
1.010	0.4988
1.100	0.4881
1.250	0.4721
1.500	0.4495

Now, with a picture, we may be able to answer the question "how does $g(x)$ behave as x approaches 1?" However, if we did not have a picture, it would be useful to generate a table of values, just as before. We then analyze $g(x)$ as x gets closer to 1 from the left and from the right. We can deduce then that

$$\lim_{x\to 1} g(x) = 0.5,$$

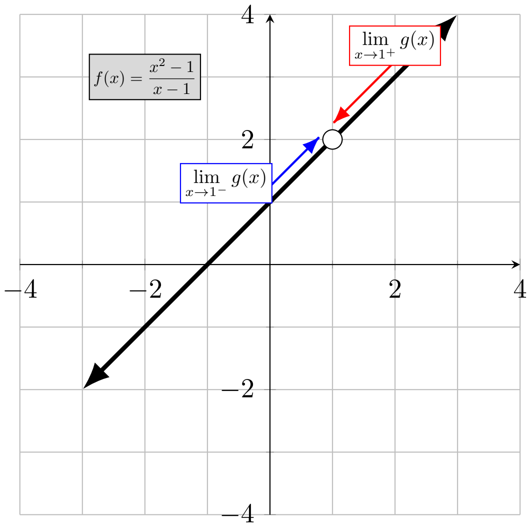
or "as x approaches 1, the value $y = g(x)$ approaches 0.5." It does not matter that $f(1)$ is undefined, or in other words, it does not have to be the case that $f(1) = 0.5$ in order for $\lim_{x\to 1} f(x) = 1$. In the eyes of the limit, we are only concerned with the function f's behavior as x gets infinitely close to 1.

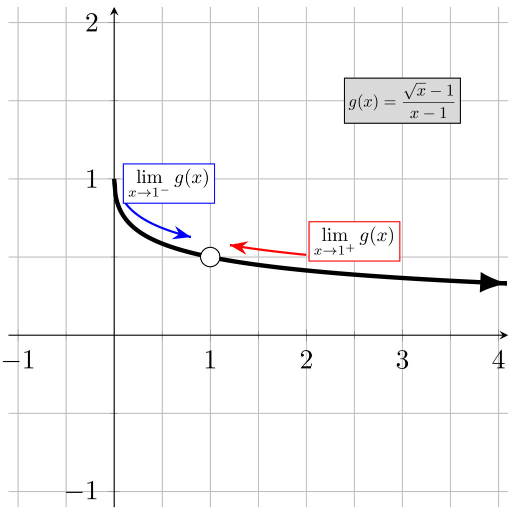
In Examples 1.9 and 1.10 we analyzed the behavior of $f(x)$ as x approached a value a from both sides. In particular, we looked at $f(x)$ for values of $x < a$ (left) that are closer and closer to a, and values of $f(x)$ for $x > a$ (right) that are closer and closer to a. While we've been talking broadly about analyzing the behavior of $f(x)$ as x approaches a, in practice we analyze the behavior of $f(x)$ as x approaches a from the left and from the right. This analysis gives rise to the one-sided limits

$$\lim_{x\to a^-} f(x) \quad \text{and} \quad \lim_{x\to a^+} f(x),$$

or the limit as x approaches a from the left of $f(x)$ and the limit as x approaches a from the right of $f(x)$ respectively.

So far we've seen a function approach the same value from the left and from the right. However, just like equations with no solutions, it may not always be the case that the limit of a function exists at a point.

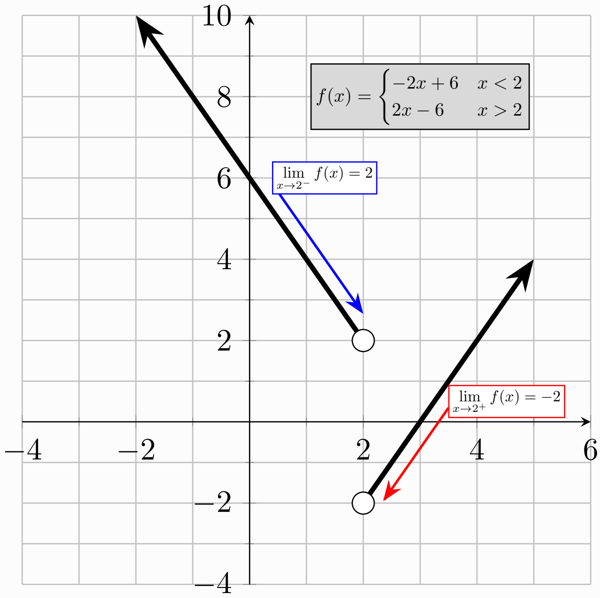




Example 1.11: Limit Does Not Exist

Consider the piecewise-defined function

$$f(x) = \begin{cases} -2x+6 & x < 2 \\ 2x-6 & x > 2 \end{cases}$$



If draw a graph of $f(x)$, we notice that there is a gap at $x = 2$; that is, to the left of $x = 2$ the function f is the line $-2x + 6$, and to the right of $x = 2$ the function f is the line $2x-6$. Notice that the function is not defined for $x=2$, in other words both segments of the piecewise function end in or start on a hole. Still, in the eyes of the limit, that is okay! Visually, we can see that

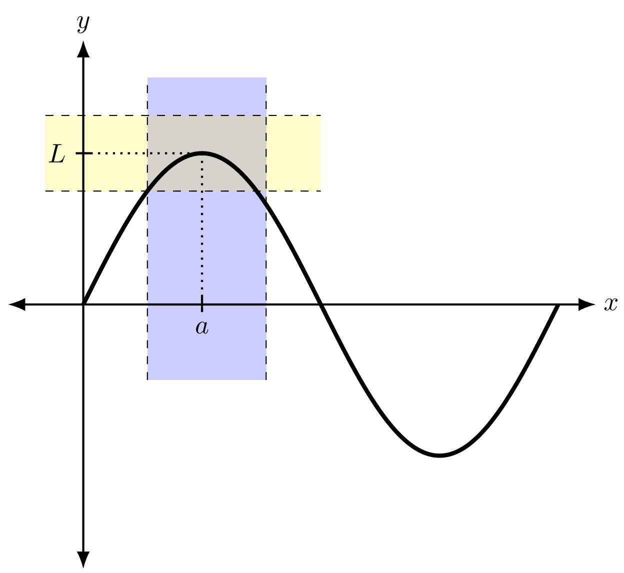
$$\lim_{x\to 2^-} f(x) = 2.$$

Analytically, if we were to make a table, then our table of values for the left-sided limit would be generated using the line $-2x+6$. In the same way,

$$\lim_{x\to 2^+} f(x) = -2.$$

Since the one-sided limits do not agree, we say the limit as x approaches 2 of $f(x)$ does not exist.

After seeing some concrete examples, let's take a moment to understand the limit of a function more abstractly. To the familiar, this view comes from the $\varepsilon - \delta$ definition of a limit. We investigate $\lim\limits_{x\to a} f(x)$, and we suspect there is some number L that $f(x)$ approaches as $x \to a$ from the left and right.



Now we can put everything we've learned thus far into a definition.

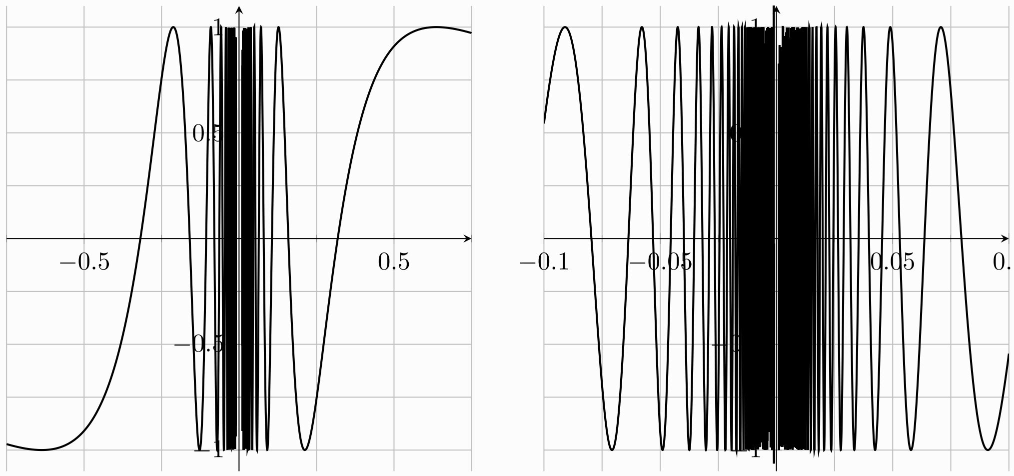
Definition 1.12: Limits

- When we write $\lim_{x \to a^-} f(x) = L$ we mean "For values of x **less than** a, as the value of x gets closer to a, the value of $f(x)$ gets closer to L." This is the left-sided limit.
- When we write $\lim_{x \to a^+} f(x) = L$ we mean "For values of x **greater than** a, as the value of x gets closer to a, the value of $f(x)$ gets closer to L." This is the right-sided limit.
- When we write $\lim_{x \to a} f(x) = L$ we mean that both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, in other words
$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$
- In the case the one-sided limits disagree, i.e. $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, we say that $\lim_{x \to a} f(x) = L$ **does not exist.** Sometimes we write DNE, but it is improper to say $\lim_{x \to a} f(x) = \text{DNE}$. We don't say "equals does not exist".

1.2.1 Limits Can Not Exist In Interesting Ways

Example 1.13: The Limit Does Not Exist – Funky ($\sin(1/x)$)

Consider the function $f(x) = \sin(1/x)$ and the limit
$$\lim_{x \to 0} f(x).$$
We ask ourselves the question: How does f behave as x gets closer to 0? In particular, we know that $f(0)$ is undefined, since $1/0$ is undefined. We also know that $\sin(\blacksquare)$ is always between -1 and 1. So, $-1 \leq \sin(1/x) \leq 1$. Furthermore, we know sine is an oscillating function; the values $f(x)$ as $x \to 0$ will bounce around between -1 and 1.



There is nothing that is restricting $\sin(1/x)$ as x approaches 0. In other words, for values of x infinitely close to 0, the function f may alternate endlessly between two values. For example, we know that $\sin(\pi/2) = 1$ and $\sin(3\pi/2) = -1$. So if we plug in the reciprocals into f:

$$f\left(\frac{2}{\pi}\right) = \sin\left(\frac{1}{\frac{2}{\pi}}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

and

$$f\left(\frac{2}{3\pi}\right) = \sin\left(\frac{1}{\frac{2}{3\pi}}\right) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

Finally, since sine is a periodic function

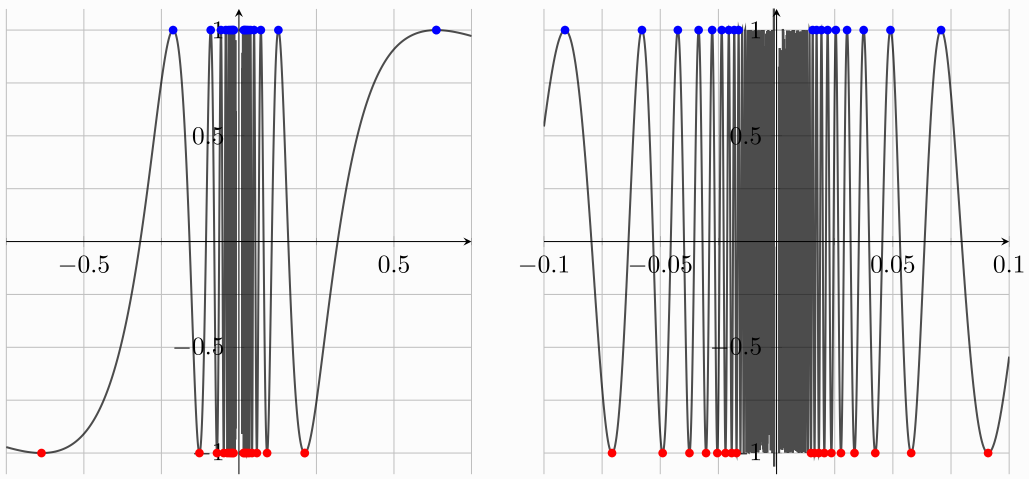
$$\sin(x) = 1 \quad \text{for} \quad \ldots -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$$

and

$$\sin(x) = -1 \quad \text{for} \quad \ldots -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$$

Simply: in one period of sine starting from $x = 0$, we have $\sin(x) = 1$ exactly once at $\pi/2$, and to get the other infinitely many points where $\sin(x) = 1$ we add or subtract multiples of 2π.

That's a lot of explanation and math, but what is it that we are really showing? We are showing that the limit doesn't exist by coming up with an infinite sequence of inputs that approach zero, but for which $\sin(x)$ does not approach a limit.



1.2.2 Infinite Limits

- Example with rational function.
- Define vertical asymptotes using limits.

1.3 Limit Laws

Limit Laws/Properties

Suppose that c is a constant, f and g are functions, and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. The limit of a **sum** is the sum of the limits.

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. The limit of a **difference** is the difference of the limits.

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3. The limit of a **constant** times a function is the constant times the limit of the function.

$$\lim_{x \to a}[cf(x)] = c[\lim_{x \to a} f(x)]$$

4. The limit of a **product** is the product of the limits.

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

5. The limit of a **quotient** is the quotient of the limits.

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

so long as $\lim_{x\to a} g(x) \neq 0$

6. The limit of a **power** of a function is the power of the limit of the function (for powers that are a positive integer $(1, 2, 3, \ldots)$.

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

7. The limit of a **root** of a function is the root of the limit of the function (for roots that are a positive integer ($\sqrt[2]{}, \sqrt[3]{}, \ldots$).

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

Remember: $\sqrt[n]{\blacksquare} = \blacksquare^{1/n}$.

1.4 Evaluating Limits – Algebraic Tricks

Ray's To-do

- Add a limit example where you have to:
  - Cancellation (why does this work... holes in a graph):

$$\lim_{x\to 2} \frac{x^2+6x+8}{x^2+10x+16}$$

  - Multiplying by a conjugate:

$$\lim_{x\to 1} \frac{\sqrt{x}+1}{(x-1)(x+2)}$$

  - Clear denominators or combine fractions:

$$\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$$

  - Analyze Rational functions around an asymptote.

$$\lim_{x\to 5} \frac{x-1}{(x-2)^2(x-5)} \quad \text{and} \quad \lim_{x\to 2} \frac{x-1}{(x-2)^2(x-5)}$$

1.5 Continuity

Ray's To-do

- Definition of continuity, from precalculus, then using calculus
  - Precalc: Can draw the graph without lifting your pen off the paper
  - Calc: A function is continuous at a point $x = a$ if $\lim_{x\to a} f(x) = f(a)$, the limit as x approaches a of $f(x)$ is the function evaluated at a. This gives us a continuity checklist:
    * Does $\lim_{x\to a} f(x)$ exist?
    * Is $f(a)$ defined?
    * Does $\lim_{x\to a} f(x) = f(a)$?
- Three main types of discontinuity with pictures
  - Removable discontinuity: hole in a graph
  - Jump discontinuity: gap in a function, like with piecewise functions.
  - Infinite discontinuity: vertical asymptotes
- Find intervals of continuity: Given a function, where is it continuous?
- Real life examples of continuity, of discontinuity
  - Outside temperature as a function of time, the earths rotational position relative to the sun as a function of time, are both continuous.
  - Something with discontinuities: lights in a room as a function of time, as lights are either on or off.
- Intermediate value theorem - If a function is continuous on an interval $[a, b]$, then $f(x)$ attains every value between $f(a)$ and $f(b)$ (look back to picture in notes.
  - Example: Temperature on antipodal points of earth. By IVT, two opposite points on earth must be the same exact temperature at some point in time throughout a day.
  - Find another more relatable example. Wobbly chair/table example seems too long winded...

2 Derivatives and Rates of Change

Ray's To-do:

- Limit definition of the derivative of $f(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

- Equation of the line tangent to a curve at a point examples - one with a trig function or two.
  * To find the equation of the line tangent to a curve $f(x)$ at a point a, first you need the **slope of the tangent line**, which is the derivative at a. In other words, find $f'(a)$. Then, you need the x and y coordinates of the point the tangent line intersects $f(x)$ at, namely $(a, f(a))$. Then either use $y = mx + b$ or $y - y_0 = m(x - x_0)$ where $m = f'(a)$ to solve for b or get the equation.
- What does it mean for a function to be differentiable? We learned three ways a function is not differentiable at a point:
  * Sharp corners: for example $f(x) = |x|$ at $x = 0$
  * Discontinuities.
  * Vertical tangent: $f(x) = \sqrt[3]{x}$ (Use power rule to see why the derivative doesn't exist at 0 and to understand why there is a vertical tangent line at 0.)
- (For Ray specifically, ignore if student) Linear approximations are good enough estimates over a short enough period of time – if a particle is moving in one direction with position function $s(t)$, then using average rate of change you may be able to predict where the particle is going to be after a very small change in time.
- Derivative Rules
  * Sum/difference rule: $(f+g)' = f' + g'$

* Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$
* Product rule: $(fg)' = f'g + fg'$. "Derivative of the first, leave second alone, plus leave first alone, derivative of the second."
  - Note for Ray: Refine rectangle intuition example.
* Quotient rule: $\frac{d}{dx}\frac{f}{g} = \frac{gf'-fg'}{(g)^2}$ "Low DHigh not High DLow, all over a LowLow."
* Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ "Derivative of the outside, leave the inside alone, times the derivative of the inside."
  - Intuition: https://math.stackexchange.com/q/62614/328970