Introducing 3-Path Domination

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Graphs - Definitions

Graph

$G = (V, E)$ is a graph G with a set of vertices $V(G)$ and set of edges $E(G)$. We will work with *simple graphs* containing no loops or multiple edges.

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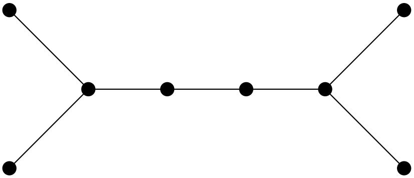

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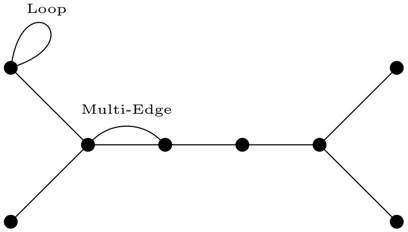

Domination

Dominating Set

The set $S \subseteq V(G)$ is a dominating set of G if for every vertex $v \in V(G)$, either $v \in S$ or v is adjacent to a vertex in S. Every $v \in S$ can be thought of as a guard that dominates adjacent rooms (vertices are rooms, edges are hallways).

Domination Number

The minimum cardinality of a dominating set, denoted $\gamma(G)$.

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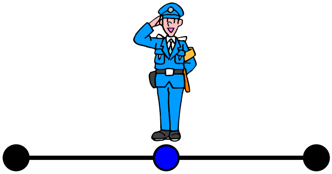
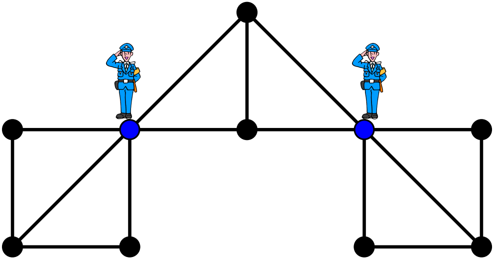


Domination

Minimum Dominating Set Example

A case where $\gamma(G) = 2$. Every guard can watch every adjacent room, including theirs.



Paired-Domination

Introduced by Haynes and Slater in 1998

Paired-Dominating Set

A dominating set where the induced subgraph on the set contains a perfect matching. In the sense of the guard analogy, every guard has another guard watching their back.

Paired-Domination Number

The minimum cardinality of a paired-dominating set, denoted $\gamma_{pr}(G)$, is always even.

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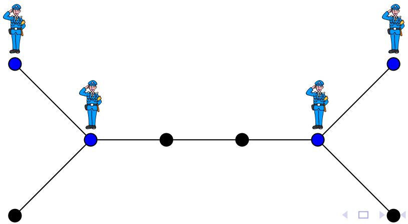

3-Path Domination

3-Path Dominating Set

Let Q_i represent a path on 3 vertices. Then we define a 3-path dominating set of G to be $S = \{Q_1, Q_2, \ldots, Q_k\}$ such that the vertex set $V(S) = V(Q_1) \cup V(Q_2) \cup \cdots \cup V(Q_k)$ is a dominating set.

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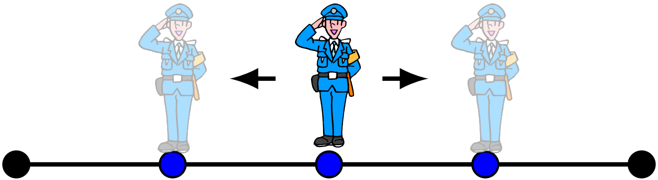

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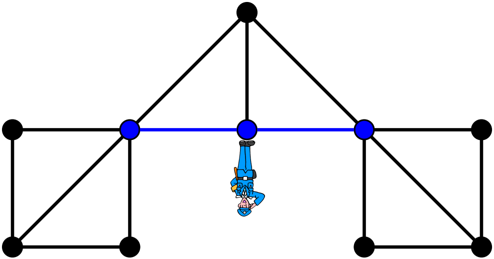

3-Path Domination Number

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The minimum number of 3-paths needed to dominate a graph G, denoted $\gamma_{P_3}(G)$.

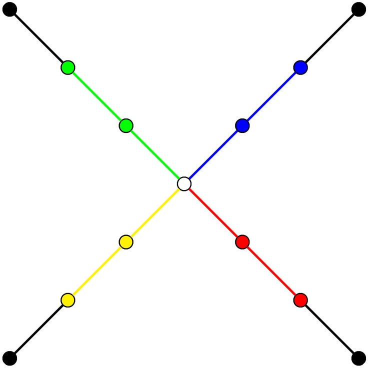
Justification

A vertex in a 3-path dominating set can be in more than one 3-path. So, the number of vertices in the set does not directly translate to the number of 3-paths.

3-Path Domination Number

Example

Below is an example of minimum 3-path dominating set.
Although 9 vertices are being used, we have 4 3-paths.



What is minimal?

Minimal

A dominating set S is said to be minimal if for a vertex $v \in S$, $S \setminus v$ is not a dominating set. A minimal dominating set is not a proper subset of any other dominating set.

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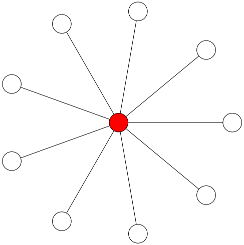

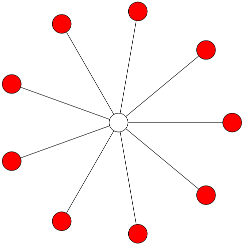

The 3-Path Domination Problem is NP-complete

Theorem

Deciding for a given graph H and positive integer K such that $3K \leq |V(H)|$, "Is $\gamma_{P_3}(H) \leq K$?" is NP-complete.

- In Haynes and Slater's paper, Paired Domination in Graphs, we get the result that paired domination is NP-complete.
- We generalized this result to show that the 3-path domination problem is also NP-complete.

Upper Bound

Lemma

For a connected graph G, there exists an edge-disjoint minimal 3-path dominating set.

Figure: Two 3-paths, $\{v_2, v_3, v_4\}$ and $\{v_3, v_4, v_5\}$ dominating the path graph P_6.

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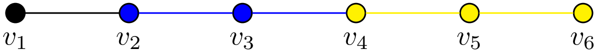

Figure: Two 3-paths, $\{v_2, v_3, v_4\}$ and $\{v_4, v_5, v_6\}$ dominating the path graph P_6.

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Lemma

For a connected graph G, there exists an edge-disjoint minimal 3-path dominating set.

Upper Bound

By the above lemma, we have $\gamma_{P_3}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor$.

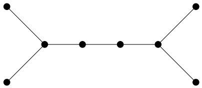
Trees

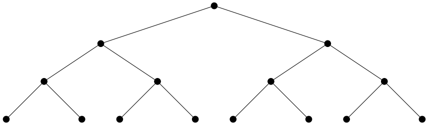
Tree

A tree T_n is an acyclic, connected graph on n vertices. Every tree has $n-1$ edges.

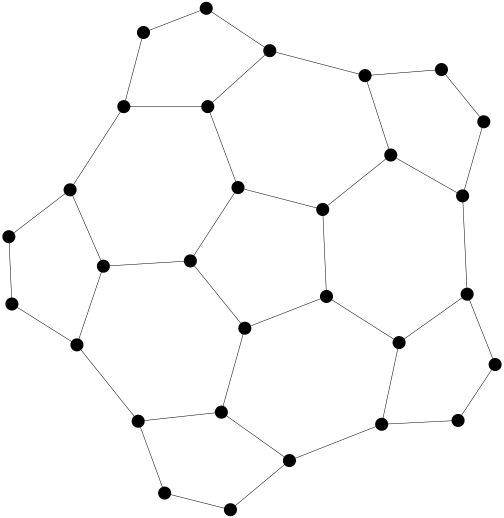
Spanning Tree

A spanning tree T_G of a graph G is a tree with vertex set $V(G)$ and edge set $E(T_G) \subseteq E(G)$.





A graph G with $n = 30$ vertices.



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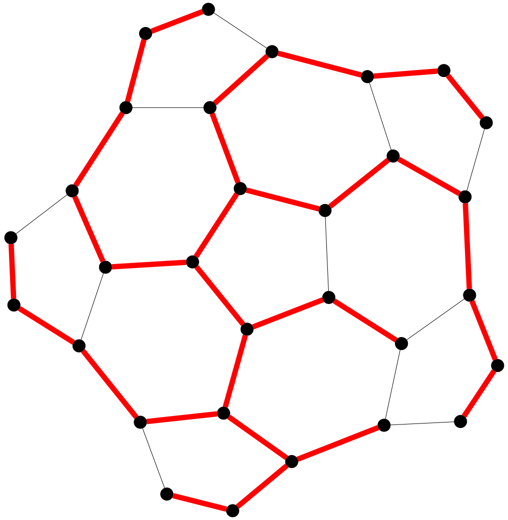
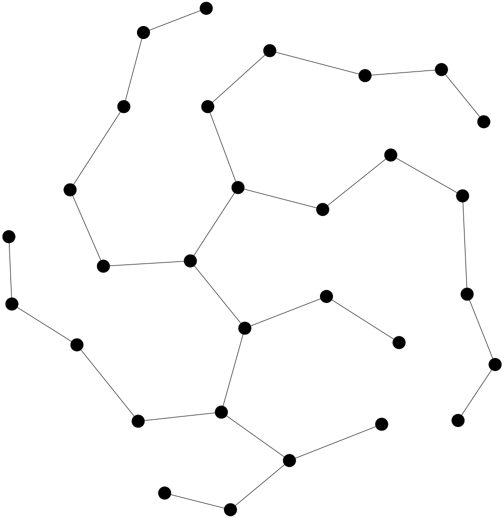


The Spanning Tree T_G of G.



A Small Example

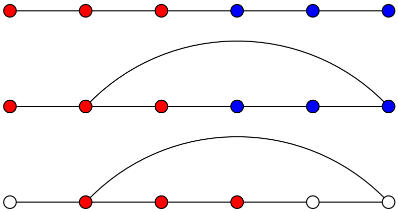


Figure: An example where the blue path is no longer needed when an edge is added.

A More Explicit Bound

Theorem

For a connected graph G on n vertices, $\gamma_{P_3}(G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof.

Let G be a connected graph on n vertices and T_G be its spanning tree. We know $\gamma_{P_3}(T_G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. In order to obtain G from T_G, we need to add our missing edges, namely $E(G) \setminus E(T_G)$. Observe, adding edges to a graph will either keep the domination number the same or cause it to decrease. So,

$$\gamma_{P_3}(G) \leq \gamma_{P_3}(T_G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

□

3-Path Domination vs. Paired Domination

Theorem

For any connected graph G on $n \geq 3$ vertices, $\gamma_{P_3}(G) \leq \frac{\gamma_{pr}(G)}{2}$.

Sketch of Proof.

Let D be a minimum paired-dominating set of G. Every pair of vertices in D has a neighbor that is not in D. So we can count every pair and a neighbor as a 3-path. So, $\gamma_{P_3}(G) \leq \frac{\gamma_{pr}(G)}{2}$. □

3-Path Domination vs. Paired Domination

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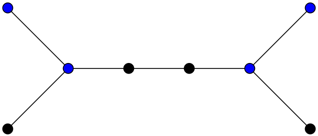

Figure: A γ_{pr}-set

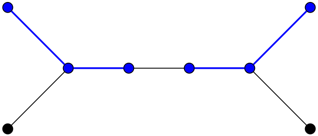

Figure: The γ_{pr}-set from above turned into a γ_{P_3}-set

Conjecture

Theorem (Haynes et al.)

For any connected graph G on $n \geq 6$ vertices and $\delta(G) \geq 2$, $\gamma_{pr}(G) \leq \frac{2n}{3}$.

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Additional Results

- Comparisons between $\gamma(G)$, $\gamma_{pr}(G)$, and $\gamma_{P_3}(G)$
- Tighter bounds for certain families of graphs, such as:
  - Caterpillar Trees
  - Graphs with a Hamiltonian Path
- Closed formulas for certain families of graphs, such as:
  - Paths or Cycle Graphs
  - Harary graphs $H_{k,n}$ where k is even
  - Banana Trees
- Algorithms for finding a minimal 3-path dominating set in trees

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