The Reconstruction Conjecture
A Brief Introduction

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Graphs - Definitions

Graph

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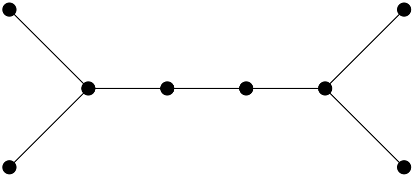

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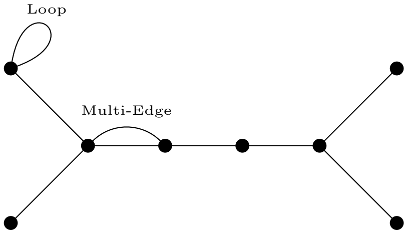

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Then A and B are isometric.

The Deck of a Graph - Definition

Definition – $D(G)$

We define the *deck* of a graph G, denoted $D(G)$, to be the collection of vertex-deleted subgraphs of G. That is, the deck is the multiset

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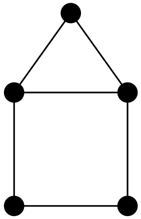
- A deck may contain several isomorphism classes. The size of an isomorphism class is its *multiplicity*.

Example - The House Graph



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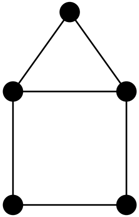

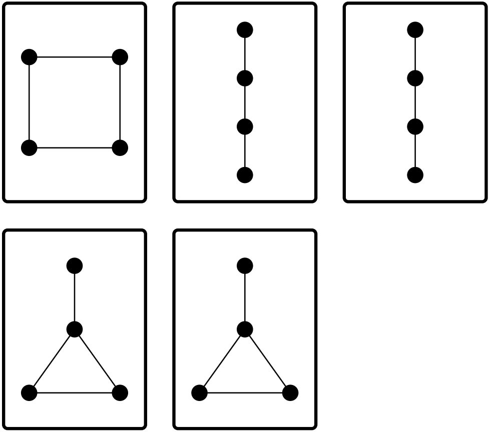

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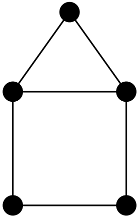

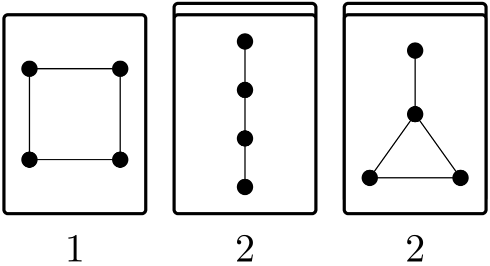

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- Vertex Connectivity: $\kappa(G) = 1 + \min_{H \in D(G)} \kappa(H)$

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A graph H with the same deck as G is said to be a *reconstruction* of G. If every reconstruction of G is isomorphic to G, then we say G is *reconstructible*.

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Reconstructible – Two Meanings

A **function** is reconstructible if it holds the same value over all reconstructions of G, whereas a **graph** is reconstructible if it is determined by its deck, up to isomorphism.

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From before...

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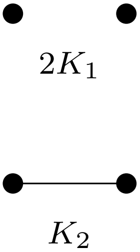
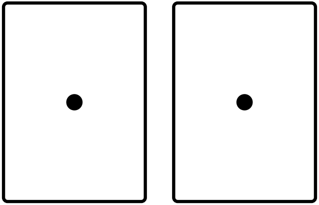
The Reconstruction Conjecture – $n \geq 3$?

Counter-example for $n = 2$

The graphs K_2 and $2K_1 \cong \overline{K_2}$ are not isomorphic, yet have the same deck.





A Note On The Legitimacy of Decks

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- For the RC, we assume any given deck is legitimate.
- Determining whether a given deck is legitimate is at least as hard as the Graph Isomorphism problem [Mansfield '80].

The Reconstruction Conjecture – Evidence?

Definition – Reconstruction Number

The *reconstruction number*, denoted $\mathrm{rn}(G)$, of a graph G is the cardinality of a smallest subdeck determining G. Note that $\mathrm{rn}(G) \geq 3$ and $\mathrm{rn}(G)$ is bounded above by quantities well below n.

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- Ally-reconstruction – How you would expect. We can choose any cards.
- Adversary-reconstruction – Evil adversary orders the deck. Whats the worst case?

The Reconstruction Conjecture – Evidence?

$G_{p,n}$

A random graph with n vertices and edges chosen randomly and independently with probability p.

Result – a.e. Graph has $\mathrm{rn}(G) = 3$ [Bollobás '90]

Let $c > 5/2$ and $(c \log n)/n \leq p = p(n) \leq 1 - (c \log n)/n$. Then a.e. $G_{p,n}$ has reconstruction number 3 for $n \geq 3$.

The Reconstruction Conjecture – Evidence?

Kelly's Lemma [Kelly '57]

Let F and G be graphs such that $|F| < |G|$. Define $s(F, G)$ to be the number of subgraphs of G isomorphic to F. Then $s(F, G)$ is reconstructible.

Proof.

$$s(F, G) = \frac{1}{|G| - |F|} \sum_{H \in D(G)} s(F, H).$$

□

The Reconstruction Conjecture – Evidence?

Immediate Consequences of Kelly's Lemma

Using the equation

$$s(F,G) = \frac{1}{|G| - |F|} \sum_{H \in D(G)} s(F,H),$$

setting $F = K_2$ we have

$$s(K_2,G) = \frac{1}{|G| - 2} \sum_{H \in D(G)} ||H|| = ||G||.$$

Hence the number of edges and degree sequence are recontructible.

Classes of Graphs

Reconstructible Classes

- Trees – for $|T| \geq 5$, $\text{rn}(T) = 3$ [Myrvold '90]... CASE WORK
- Disconnected Graphs – find largest component. For each component, there is a card containing it.
- Regular graphs – all vertices have the same degree. Use consequences of Kelly's Lemma.

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Non-reconstructible graphs

- Directed graphs
- Infinite graphs – Take an infinite tree T and $2T$.
- Hypergraphs – 3-uniform hypergraphs [Kocay '87]
- ... among others