Determinants of Simple Theta Curves

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Theta Curves

Definition – Theta Curve

A *theta curve* ϑ is an embedding of a θ-graph in the three-sphere, up to equivalence by ambient isotopy.

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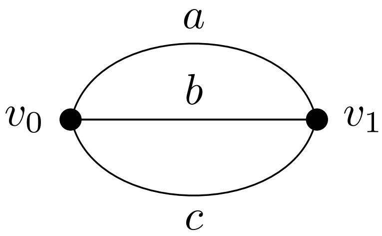

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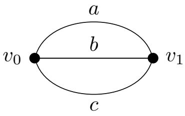

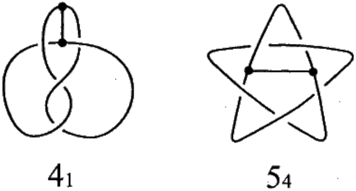

Moriuchi, 2009. Table 1.

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Definition – Constituent Knot

Every theta curve contains three *constituent knots* K_{ij} formed by taking pairs of edges $i, j \in \{a, b, c\}$.

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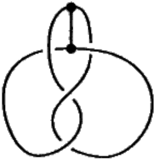

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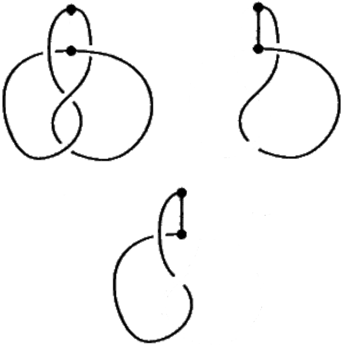

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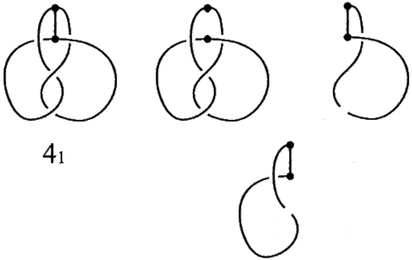

Simple Theta

A theta curve with a constituent unknot is called *simple*.

Strongly Invertible Knots

SIK (Sakuma, 1985)

A knot K in S^3 is *strongly invertible* if there is an orientation-preserving involution h on S^3 such that $h(K) = K$ and $\mathrm{Fix}(h)$ is a circle intersecting K in two points.

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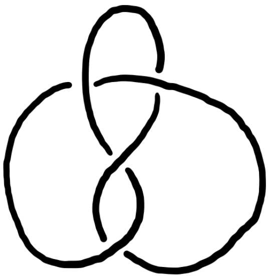

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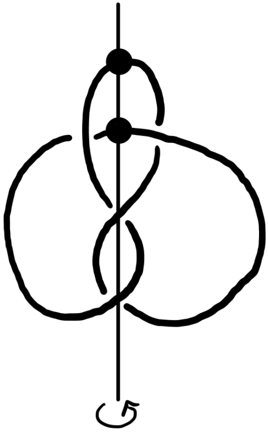

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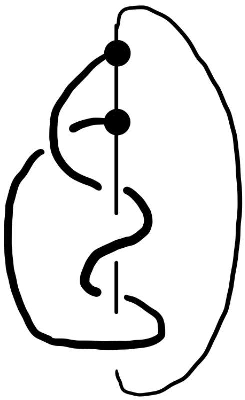
*There is a correspondence between simple ϑ and (K, h), in particular $\vartheta = K/h \cup \mathrm{Fix}(h)$



The Determinant

Determinant of a Knot

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$$\det(K) = \Delta_K(-1) = \det(V + V^T) = |H_1(\Sigma_2(S^3, K))|$$

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How do we define a determinant for ϑ?

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Σ_ϑ can be visualized by iterating a branched double cover construction.

$$\Sigma_\vartheta \cong \Sigma_2(\Sigma_2(S^3, K_{ac}), \tilde{e}_b).$$

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For ϑ with corresponding strongly invertible knot (K, h), $\det(\vartheta) = \det((K, h))$.

Main Theorem

Theorem (EIM, 2022)

If ϑ is a simple theta curve with constituent knots K_{ab}, K_{ac}, K_{bc} then

$$\det(\vartheta) = \det(K_{ab}) \cdot \det(K_{ac}) \cdot \det(K_{bc}).$$

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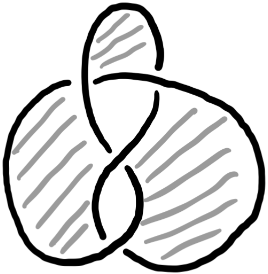

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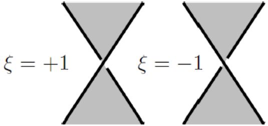

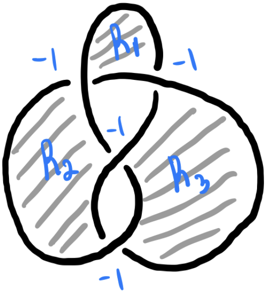

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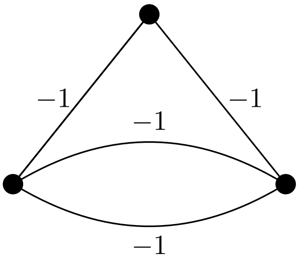

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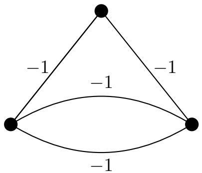

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix}$$

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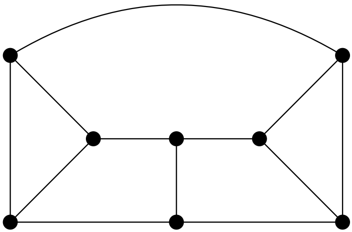

Ciucu, Yan, Zhang 2005
Yan, Zhang, 2009

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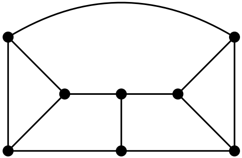

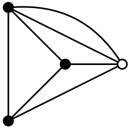

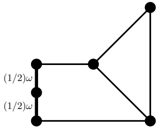

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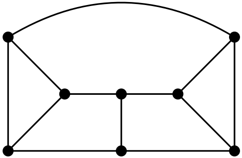

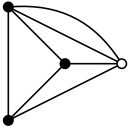

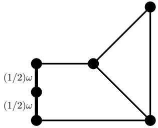

Open Questions

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- Can a spanning tree enumeration strategy can be used to calculate $\det(\vartheta)$ for non-simple theta curves or other Klein graphs (3-Hamiltonian, trivalent)?

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Thank you!

M. Elpers, R. Ibrahim, A. H. Moore, Determinants of simple theta curves and symmetric graphs, 2022. arXiv.2211.00626.

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Extra

- Σ_2: $\pi_1(X, x) \to H_1(X; \mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z}_2$
- Σ_ϑ: $\pi_1(Y, x) \to H_1(Y; \mathbb{Z}) \to \mathbb{Z}_2 \times \mathbb{Z}_2$