The Structure of Graphs With Independence Number 2

Rayan Ibrahim

VCU Discrete Mathematics Seminar
February 9, 2022

College of Humanities and Sciences
Mathematics and Applied Mathematics

Introduction

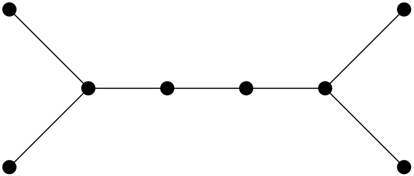
Definition – Graph

$G = (V, E)$ is a graph with a set of vertices $V(G)$ and edges $E(G)$. We will work with *finite simple graphs*; graphs that do not contain loops or multiple edges.

Introduction

Definition – Graph

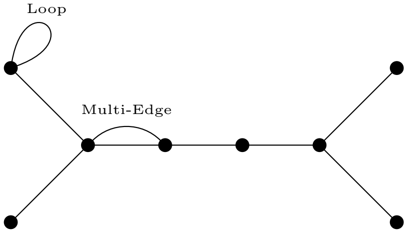
$G = (V, E)$ is a graph with a set of vertices $V(G)$ and edges $E(G)$. We will work with *finite simple graphs*; graphs that do not contain loops or multiple edges.



Introduction

Definition – Graph

$G = (V, E)$ is a graph with a set of vertices $V(G)$ and edges $E(G)$. We will work with *finite simple graphs*; graphs that do not contain loops or multiple edges.



Introduction

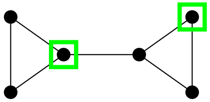
Definition – Independence

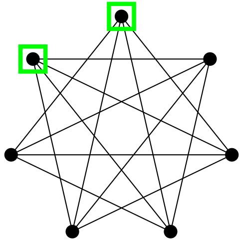
An *independent set* $S \subseteq V(G)$ is a set of pairwise non-adjacent vertices. The *independence number* of a graph G, denoted $\alpha(G)$, is the maximum size of an independent set. A graph with independence number 2 will be referred to as an alpha-two graph.

Introduction

Definition – Independence

An *independent set* $S \subseteq V(G)$ is a set of pairwise non-adjacent vertices. The *independence number* of a graph G, denoted $\alpha(G)$, is the maximum size of an independent set. A graph with independence number 2 will be referred to as an alpha-two graph.





Motivation – Why study alpha-two graphs?

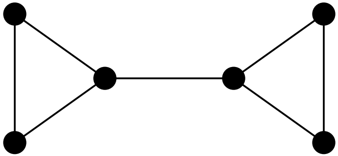
Definition – Chromatic Number

A proper coloring of a graph G assigns to each vertex of G a color so that no two adjacent vertices share the same color. The *chromatic number*, $\chi(G)$, is the minimum number of colors needed for a proper coloring.

Motivation – Why study alpha-two graphs?

Definition – Chromatic Number

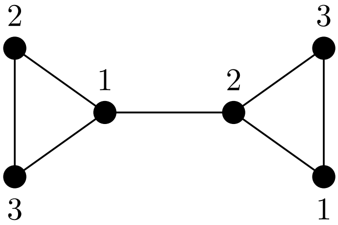
A proper coloring of a graph G assigns to each vertex of G a color so that no two adjacent vertices share the same color. The *chromatic number*, $\chi(G)$, is the minimum number of colors needed for a proper coloring.



Motivation – Why study alpha-two graphs?

Definition – Chromatic Number

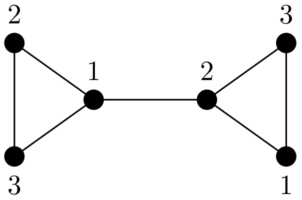
A proper coloring of a graph G assigns to each vertex of G a color so that no two adjacent vertices share the same color. The *chromatic number*, $\chi(G)$, is the minimum number of colors needed for a proper coloring.



Motivation – Why study alpha-two graphs?

Definition – Chromatic Number

A proper coloring of a graph G assigns to each vertex of G a color so that no two adjacent vertices share the same color. The *chromatic number*, $\chi(G)$, is the minimum number of colors needed for a proper coloring.



Observation

$\omega(G) \leq \chi(G)$. Need at least as many colors as the size of the largest clique.

Motivation – Why study alpha-two graphs?

Chromatic & Clique Number

It's not always the case that $\omega(G) = \chi(G)$.

Motivation – Why study alpha-two graphs?

Chromatic & Clique Number

It's not always the case that $\omega(G) = \chi(G)$.

- The gap between $\omega(G)$ and $\chi(G)$ can be arbitrarily large.

Motivation – Why study alpha-two graphs?

Chromatic & Clique Number

It's not always the case that $\omega(G) = \chi(G)$.

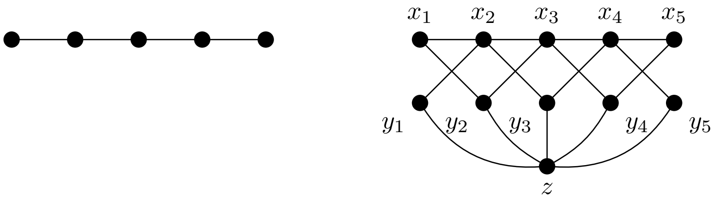
- The gap between $\omega(G)$ and $\chi(G)$ can be arbitrarily large.
- One way to show – Mycielski Construction.

Motivation – Why study alpha-two graphs?

Chromatic & Clique Number

It's not always the case that $\omega(G) = \chi(G)$.

- The gap between $\omega(G)$ and $\chi(G)$ can be arbitrarily large.
- One way to show – Mycielski Construction.
  - Take a copy of triangle-free G.
  - Add a vertex z and a vertex y_i for every $x_i \in V(G)$.
  - For each i, add edges zy_i.
  - For each edge x_ix_j add edges x_iy_j and x_jy_i.



Motivation – Why study alpha-two graphs?

Definition – Graph Minor

Let G and H be graphs, $|V(H)| \leq |V(G)|$, and let $V(H) = \{v_1, v_2, \ldots, v_k\}$. We say G contains H as a minor if there are disjoint subsets $V_1, V_2, \ldots, V_k$ of $V(G)$ satisfying the following:

- $G[V_i]$ is connected.
- If $v_i v_j \in E(H)$, then there is an edge between V_i and V_j.

Motivation – Why study alpha-two graphs?

Definition – Graph Minor

Let G and H be graphs, $|V(H)| \leq |V(G)|$, and let $V(H) = \{v_1, v_2, \ldots, v_k\}$. We say G contains H as a minor if there are disjoint subsets $V_1, V_2, \ldots, V_k$ of $V(G)$ satisfying the following:

- $G[V_i]$ is connected.
- If $v_i v_j \in E(H)$, then there is an edge between V_i and V_j.

Alternatively, one can say H can be obtained from G by a sequence of edge contractions and deletions.

Motivation – Why study alpha-two graphs?

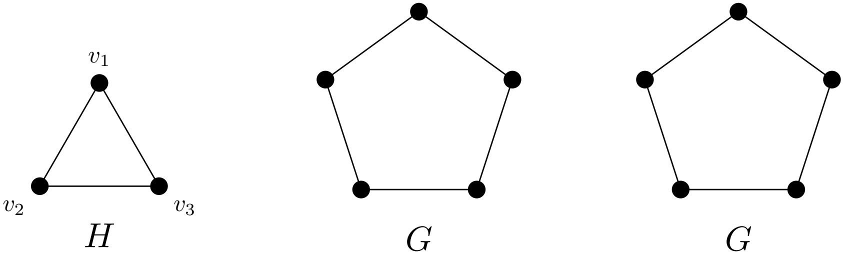
Definition – Graph Minor

Let G and H be graphs, $|V(H)| \leq |V(G)|$, and let $V(H) = \{v_1, v_2, \ldots, v_k\}$. We say G contains H as a minor if there are disjoint subsets $V_1, V_2, \ldots, V_k$ of $V(G)$ satisfying the following:

- $G[V_i]$ is connected.
- If $v_i v_j \in E(H)$, then there is an edge between V_i and V_j.

Alternatively, one can say H can be obtained from G by a sequence of edge contractions and deletions. We use the notation $G \succeq H$ to denote "G contains H as a minor", or G is contractible to H.

Graph Minors – An Example



Motivation – Hadwiger's Conjecture

Conjecture 1.0 [Hadwiger, 1946]

Let G be a graph. Then $G \succeq K_{\chi(G)}$.

Motivation – Hadwiger's Conjecture

Conjecture 1.0 [Hadwiger, 1946]

Let G be a graph. Then $G \succeq K_{\chi(G)}$.

- True for $1 \leq \chi(G) \leq 6$, otherwise open in general.

Motivation – Hadwiger's Conjecture

Conjecture 1.0 [Hadwiger, 1946]

Let G be a graph. Then $G \succeq K_{\chi(G)}$.

- True for $1 \leq \chi(G) \leq 6$, otherwise open in general.

Conjecture 1.1 [Plummer, Stiebitz, Toft, 2005]

If G is an alpha-two graph of order n, then $G \succeq K_{\lceil n/2 \rceil}$.

Motivation – Hadwiger's Conjecture

Conjecture 1.0 [Hadwiger, 1946]

Let G be a graph. Then $G \succeq K_{\chi(G)}$.

- True for $1 \leq \chi(G) \leq 6$, otherwise open in general.

Conjecture 1.1 [Plummer, Stiebitz, Toft, 2005]

If G is an alpha-two graph of order n, then $G \succeq K_{\lceil n/2 \rceil}$.

Conjecture 1.2 [Seymour, 2016]

If G is an alpha-two graph of order n, then $G \succeq K_{\lceil cn \rceil}$ for $c > 1/3$.

Some Structural Results

Definition – Diameter

The *diameter* of a graph G is the size of the longest shortest path between two vertices, i.e. $\max_{x,y \in V(G)} d(x,y)$.

Some Structural Results

Definition – Diameter

The *diameter* of a graph G is the size of the longest shortest path between two vertices, i.e. $\max_{x,y \in V(G)} d(x, y)$.

- If G is alpha-two, then $\operatorname{diam}(G) \leq 3$.

Some Structural Results

Definition – Diameter

The *diameter* of a graph G is the size of the longest shortest path between two vertices, i.e. $\max_{x,y \in V(G)} d(x,y)$.

- If G is alpha-two, then $\mathrm{diam}(G) \leq 3$.

Definition – Biclique

Let G be a graph with vertex set V. We call G a *biclique* if V can be partitioned into two sets A and B such that $G[A]$ and $G[B]$ are cliques.

Structural Results – Diameter 3

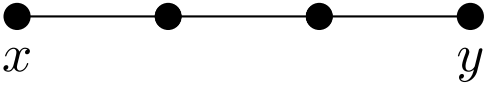
Theorem [Ibrahim, 2021]

Let G be a diameter 3 graph. If G is alpha-two, then G is a biclique.

Structural Results – Diameter 3

Theorem [Ibrahim, 2021]

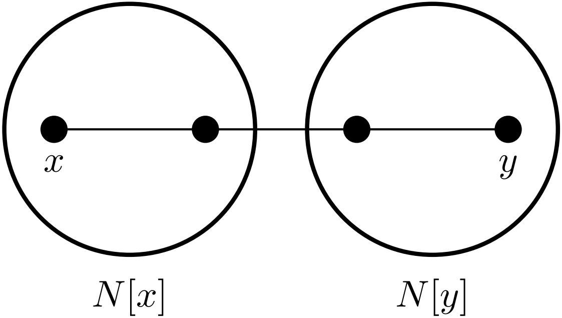
Let G be a diameter 3 graph. If G is alpha-two, then G is a biclique.



Structural Results – Diameter 3

Theorem [Ibrahim, 2021]

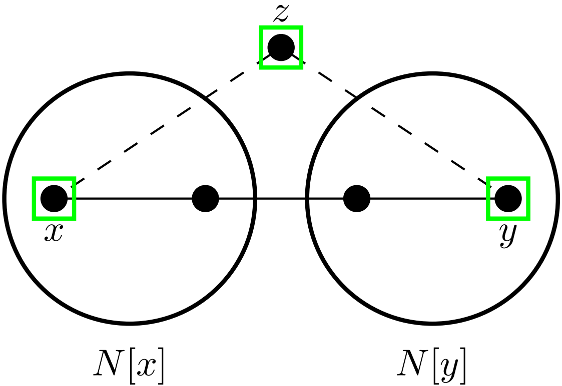
Let G be a diameter 3 graph. If G is alpha-two, then G is a biclique.



Structural Results – Diameter 3

Theorem [Ibrahim, 2021]

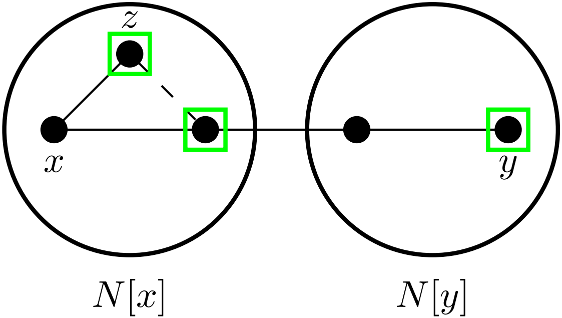
Let G be a diameter 3 graph. If G is alpha-two, then G is a biclique.



Structural Results – Diameter 3

Theorem [Ibrahim, 2021]

Let G be a diameter 3 graph. If G is alpha-two, then G is a biclique.



Structural Results – Diameter 2

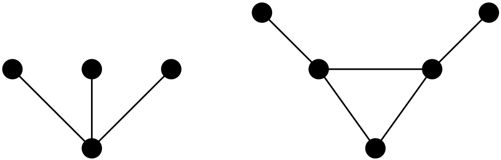
Definition – Claw-Bull-free

A *claw* is a graph with one vertex x and three pendants connected to x. A *bull* is a triangle with two pendants u, v where u and v have different neighbors.

Structural Results – Diameter 2

Definition – Claw-Bull-free

A *claw* is a graph with one vertex x and three pendants connected to x. A *bull* is a triangle with two pendants u, v where u and v have different neighbors. A graph is *claw-bull-free* if it does not contain the claw or the bull as an **induced** subgraph.



Structural Results – Diameter 2

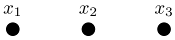
Theorem [Ibrahim, 2021]

Let G be a diameter 2 graph of order $n \geq 6$. If G is claw-bull-free, then G is alpha-two.

Structural Results – Diameter 2

Theorem [Ibrahim, 2021]

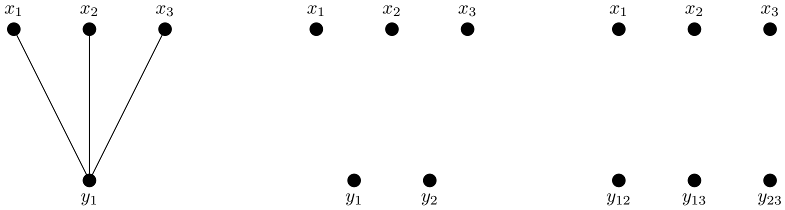
Let G be a diameter 2 graph of order $n \geq 6$. If G is claw-bull-free, then G is alpha-two.



Structural Results – Diameter 2

Theorem [Ibrahim, 2021]

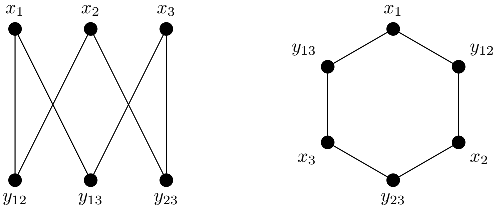
Let G be a diameter 2 graph of order $n \geq 6$. If G is claw-bull-free, then G is alpha-two.



Structural Results – Diameter 2

Theorem [Ibrahim, 2021]

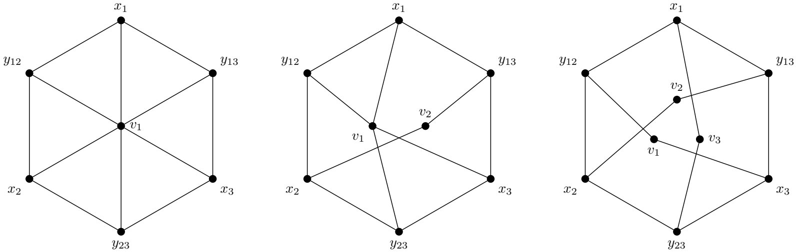
Let G be a diameter 2 graph of order $n \geq 6$. If G is claw-bull-free, then G is alpha-two.



Structural Results – Diameter 2

Theorem [Ibrahim, 2021]

Let G be a diameter 2 graph of order $n \geq 6$. If G is claw-bull-free, then G is alpha-two.



Structural Results – Lovász Theta

Definition – Clique Covering Number

A *clique cover* of a graph G is a partition of $V(G)$ such that each part induces a clique. The *clique cover number*, denoted $\bar{\chi}(G)$, is a clique cover of minimum size. Note: $\chi(\bar{G}) = \bar{\chi}(G)$.

Structural Results – Lovász Theta

Definition – Clique Covering Number

A *clique cover* of a graph G is a partition of $V(G)$ such that each part induces a clique. The *clique cover number*, denoted $\bar{\chi}(G)$, is a clique cover of minimum size. Note: $\chi(\bar{G}) = \bar{\chi}(G)$.

Properties – Lovász Number $\vartheta(G)$ [Lovász 1979]

- Real Number

Structural Results – Lovász Theta

Definition – Clique Covering Number

A *clique cover* of a graph G is a partition of $V(G)$ such that each part induces a clique. The *clique cover number*, denoted $\bar{\chi}(G)$, is a clique cover of minimum size. Note: $\chi(\bar{G}) = \bar{\chi}(G)$.

Properties – Lovász Number $\vartheta(G)$ [Lovász 1979]

- Real Number
- Efficiently computable in polynomial time [Grötschel, Lovász, Schrijver, 1981].

Structural Results – Lovász Theta

Sandwich Theorem [Knuth, 1994]

The efficiently computable $\vartheta(G)$ is “sandwiched” between two known hard problems:

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G)$$

Structural Results – Lovász Theta

Sandwich Theorem [Knuth, 1994]

The efficiently computable $\vartheta(G)$ is "sandwiched" between two known hard problems:

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G)$$

Definition – Perfect Graph

A graph G is *perfect* if for every induced subgraph H of G, $\alpha(H) = \bar{\chi}(H)$. Since the complement of a perfect graph is perfect, we can also say $\omega(H) = \chi(H)$.

Structural Results – Lovász Theta

Strong Perfect Graph Theorem [C, R, S, T, 2006]

An odd *hole* (*antihole)* is an induced cycle (complement of cycle) of length at least 5. A graph is perfect if and only if it does not contain an induced odd hole or odd antihole.

Structural Results – Lovász Theta

Theorem [Ibrahim, Larson, 2021]

If $\alpha = \vartheta = 2$ then G is perfect.

Structural Results – Lovász Theta

Theorem [Ibrahim, Larson, 2021]

If $\alpha = \vartheta = 2$ then G is perfect.

Idea: Use the SPGT. Rule out odd holes/antiholes.

- $\alpha = 2 \implies G$ does not contain an induced C_k, for odd $k > 5$.
- $\vartheta = 2 \implies G$ does not contain an induced C_5, since $\vartheta(C_5) > 2$ [Lovász 1979].

Structural Results – Lovász Theta

Theorem [Ibrahim, Larson, 2021]

If $\alpha = \vartheta = 2$ then G is perfect.

Idea: Use the SPGT. Rule out odd holes/antiholes.

- $\alpha = 2 \implies G$ does not contain an induced C_k, for odd $k > 5$.
- $\vartheta = 2 \implies G$ does not contain an induced C_5, since $\vartheta(C_5) > 2$ [Lovász 1979].
- $\vartheta(\bar{C}_k) > 2$, k odd.

Structural Results – Lovász Theta

Theorem [Ibrahim, Larson, 2021]

If $\alpha = \vartheta = 2$ then G is perfect.

Idea: Use the SPGT. Rule out odd holes/antiholes.

- $\alpha = 2 \implies G$ does not contain an induced C_k, for odd $k > 5$.
- $\vartheta = 2 \implies G$ does not contain an induced C_5, since $\vartheta(C_5) > 2$ [Lovász 1979].
- $\vartheta(\bar{C}_k) > 2$, k odd.

Properties of ϑ

1. If H is an induced subgraph of G then $\vartheta(H) \leq \vartheta(G)$.
2. If a graph is vertex transitive: $\vartheta(G)\vartheta(\bar{G}) = n$.
3. If n is odd: $\vartheta(C_n) = \frac{n\cos(\pi/n)}{1+\cos(\pi/n)}$.

Structural Results – Lovász Theta

Observation

Let G be a graph with $\alpha(G) = 2$. Then G is a biclique if and only if G is perfect.

Structural Results – Lovász Theta

Observation

Let G be a graph with $\alpha(G) = 2$. Then G is a biclique if and only if G is perfect.

Proof:

Structural Results – Lovász Theta

Observation

Let G be a graph with $\alpha(G) = 2$. Then G is a biclique if and only if G is perfect.

Proof: Complement of a biclique is bipartite.

Future Work

Conjecture 1.2 [Seymour, 2016]

If G is an alpha-two graph of order n, then $G \succeq K_{\lceil cn \rceil}$ for $c > 1/3$.

Future Work

Conjecture 1.2 [Seymour, 2016]

If G is an alpha-two graph of order n, then $G \succeq K_{\lceil cn \rceil}$ for $c > 1/3$.

- Plummer, Stiebitz, Toft offhandedly: $c = 2/5$ from contracting C_5's

Future Work

Conjecture 1.2 [Seymour, 2016]

If G is an alpha-two graph of order n, then $G \succeq K_{\lceil cn \rceil}$ for $c > 1/3$.

- Plummer, Stiebitz, Toft offhandedly: $c = 2/5$ from contracting C_5's
- Graduate Student: $c = 3/8$ using existing results and a minimum degree argument.