New Results on Bootstrap Percolation

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VCU Discrete Math Seminar
November 1, 2023

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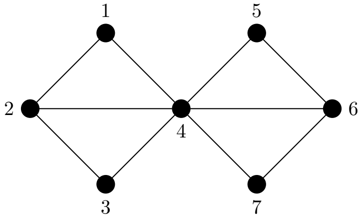

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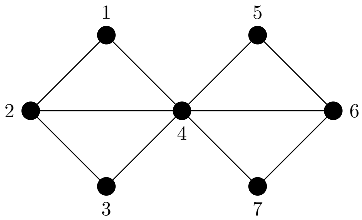

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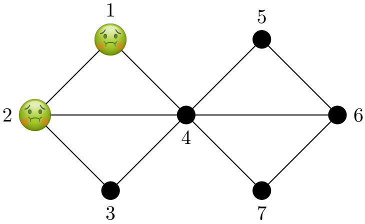

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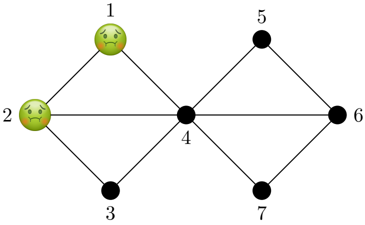

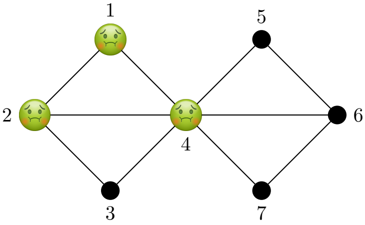

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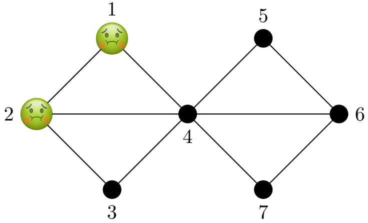

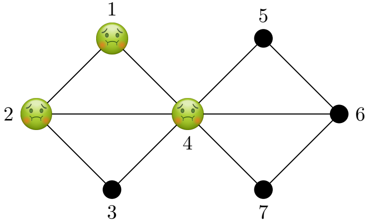

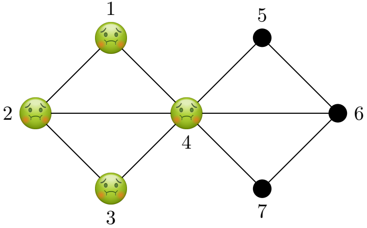

$$A_0 = \{1, 2\}$$

$$\langle A_0 \rangle = \{1, 2, 3, 4\}$$

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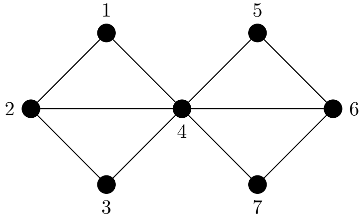

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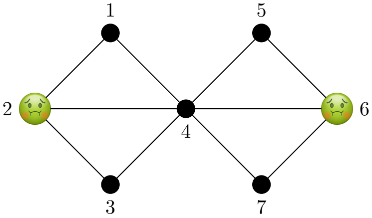

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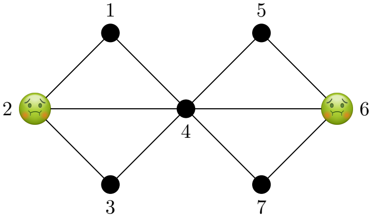

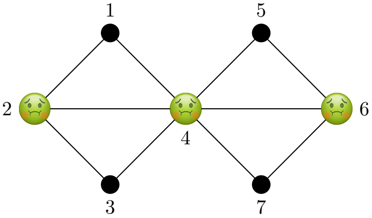

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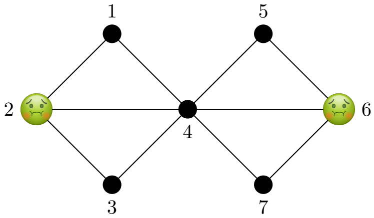

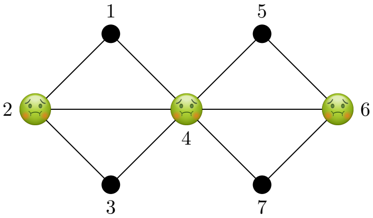

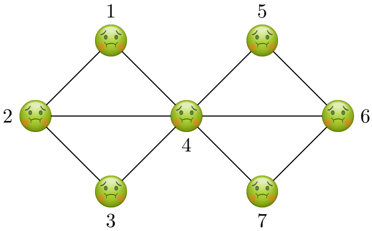

$$A_0 = \{2, 6\}$$

$$\langle A_0 \rangle = V(G)$$

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Choosing A_0

Early models incorporate randomness; initial infected vertices are selected with probability p.

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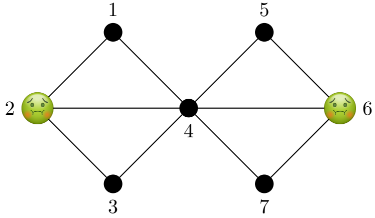

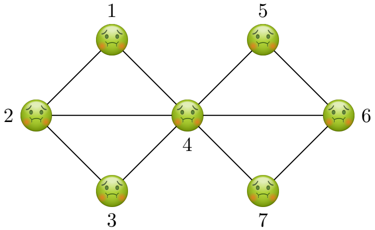

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- ... several others

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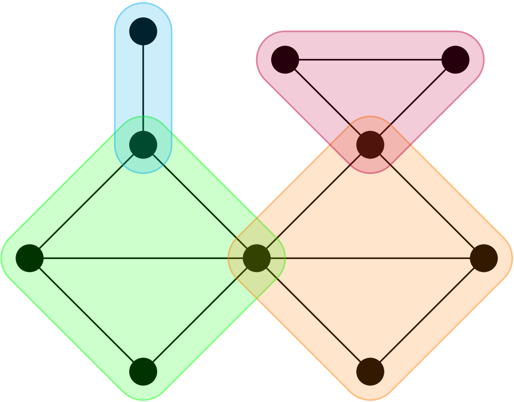

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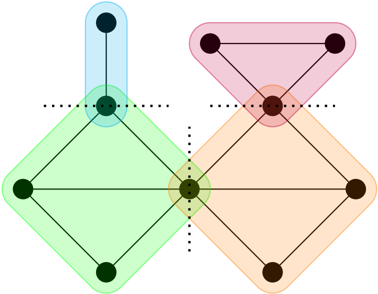

Observations

- Blocks intersect in a cut vertex.
- Blocks are 2-connected, or K_2.

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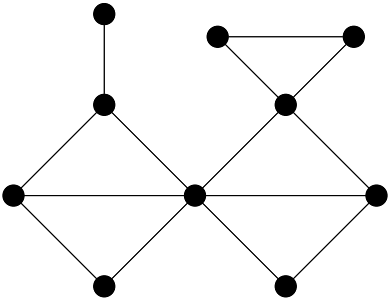

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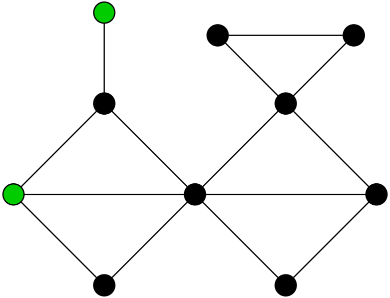

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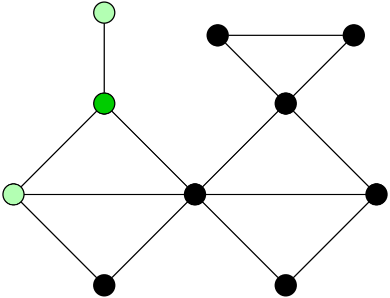

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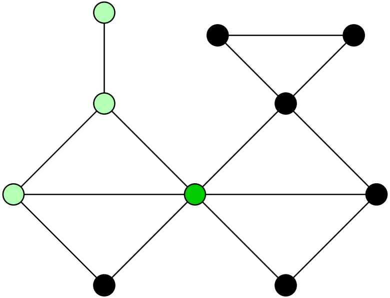

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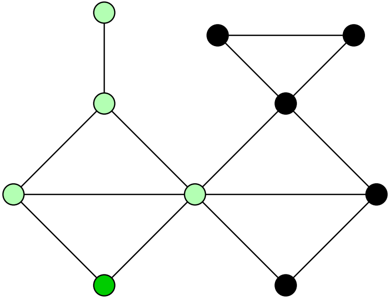

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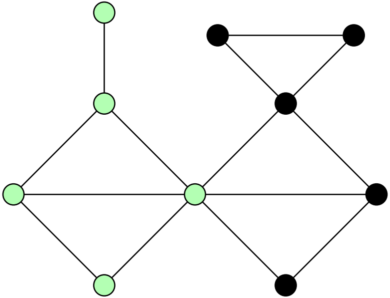

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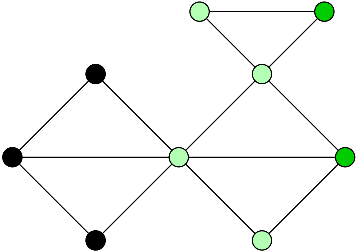

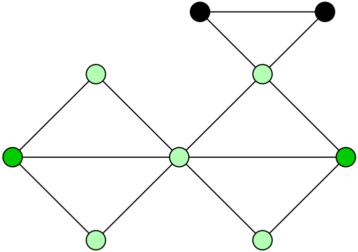

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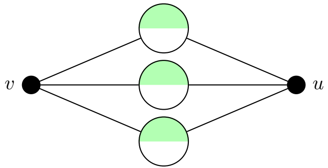

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- There are at most r components, each corresponding to a block.

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Question: What's the next best (or interesting) upper bound, and what achieves that?

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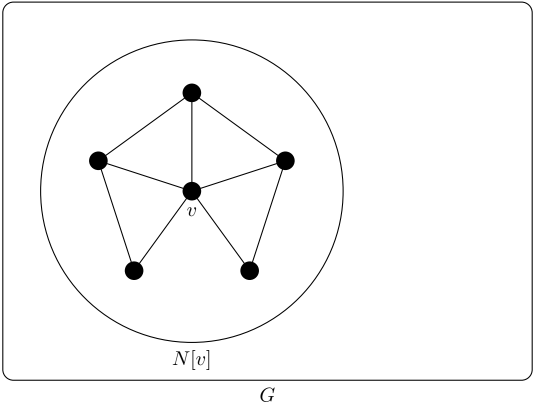

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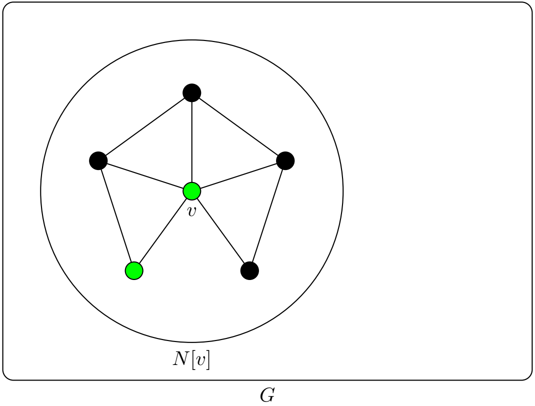

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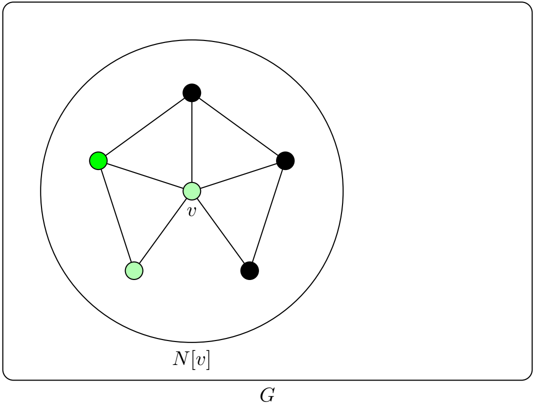

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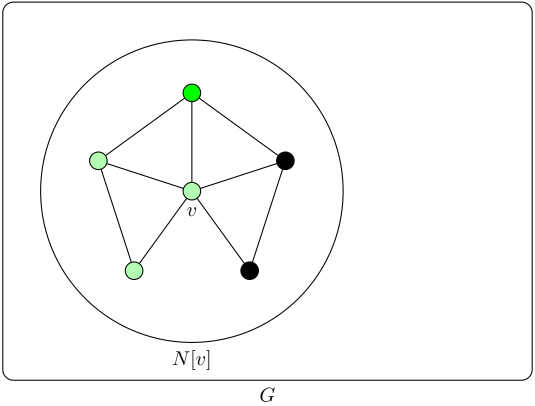

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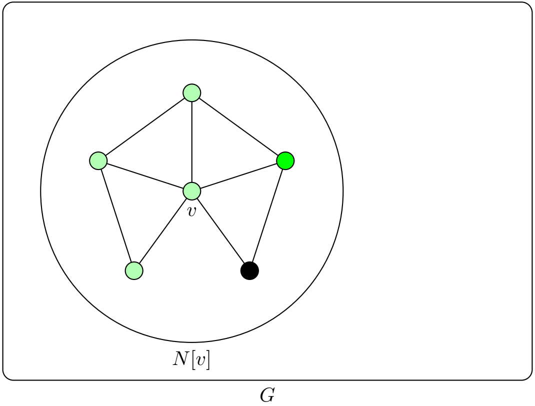

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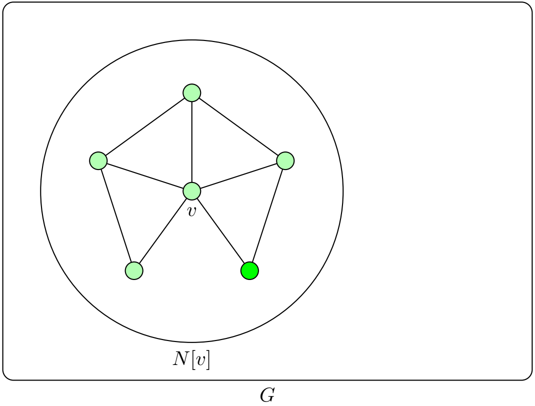

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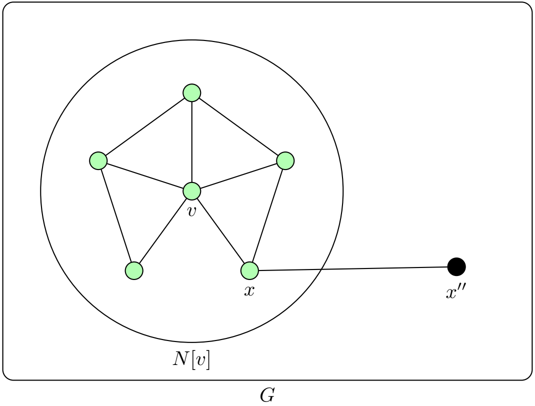

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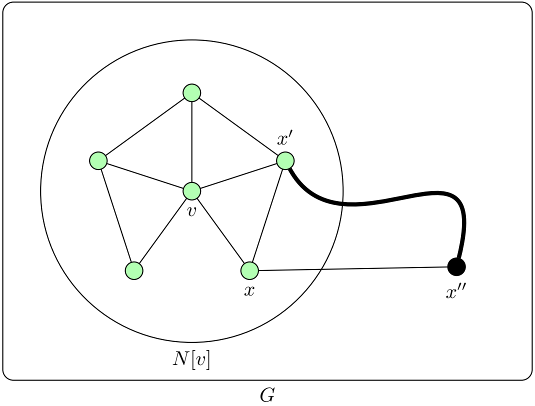

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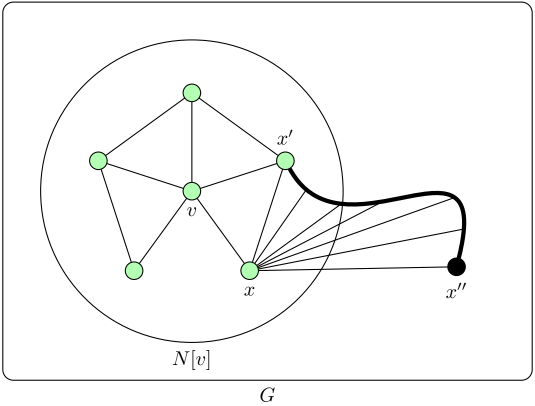

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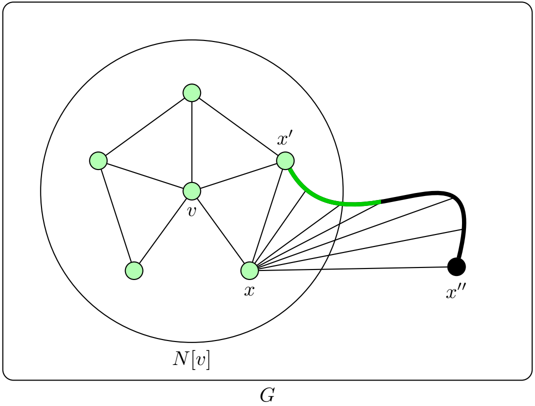

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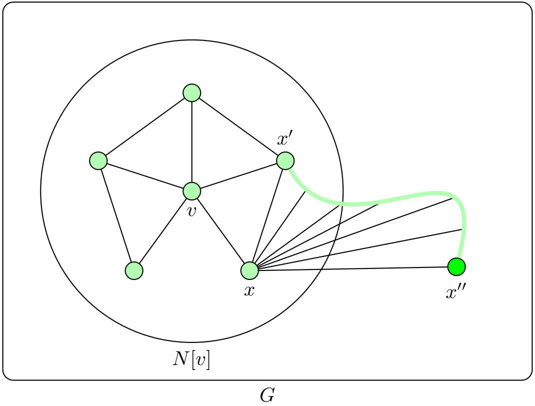

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If G is locally connected, then G is 2-BG. In particular if G does not have a leaf, then ***any pair of adjacent vertices percolates.***

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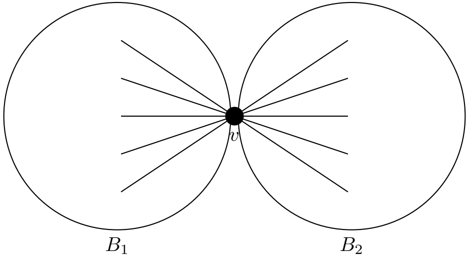

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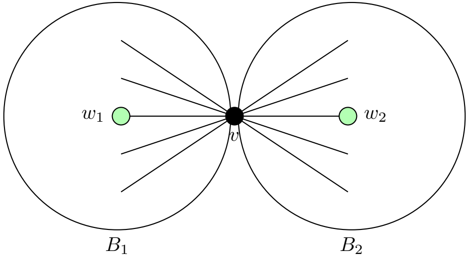

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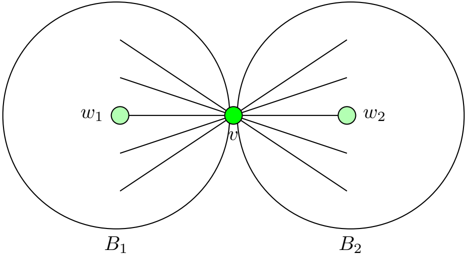

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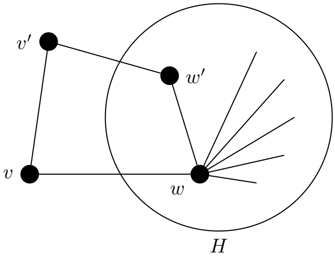

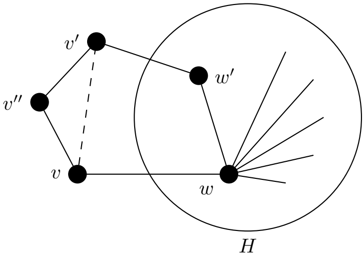

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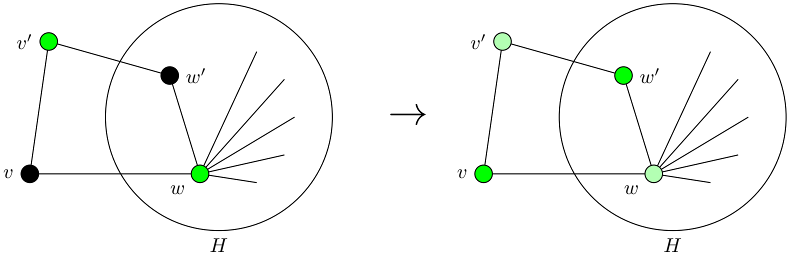

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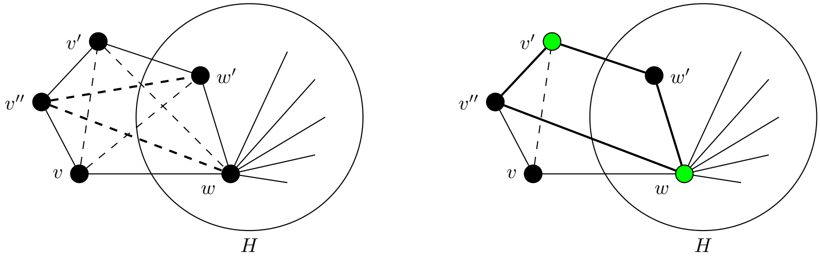

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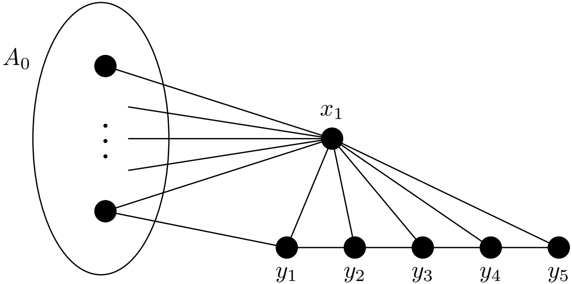

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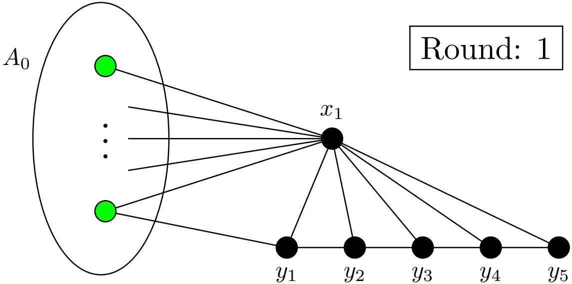

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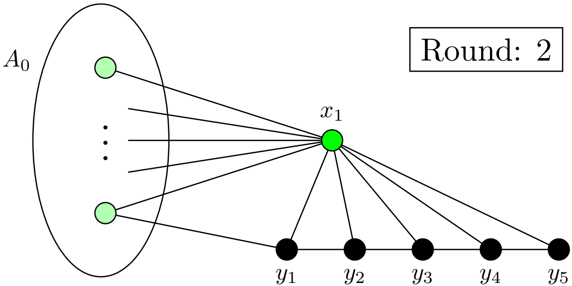

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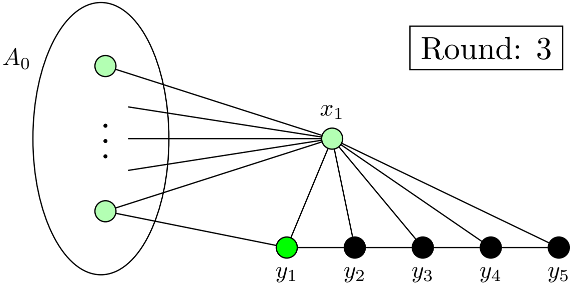

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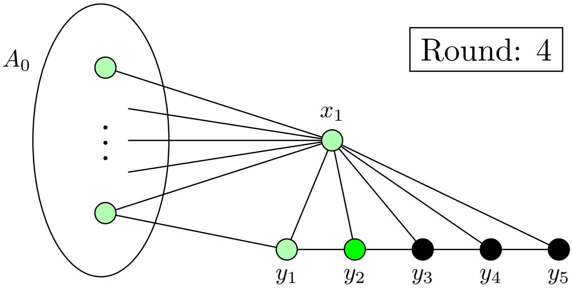

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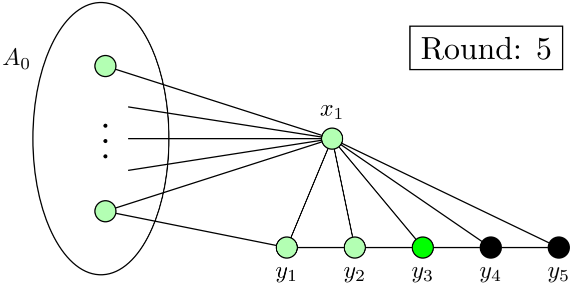

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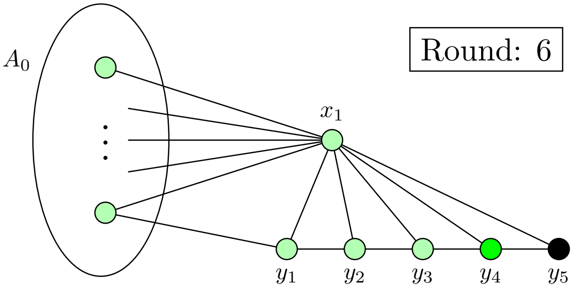

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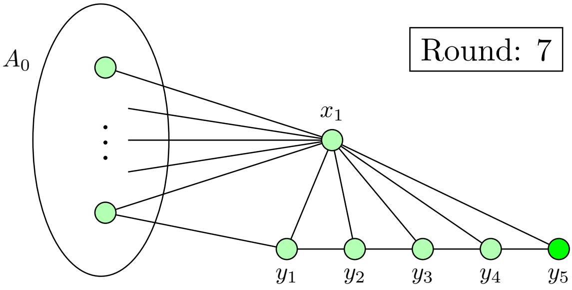

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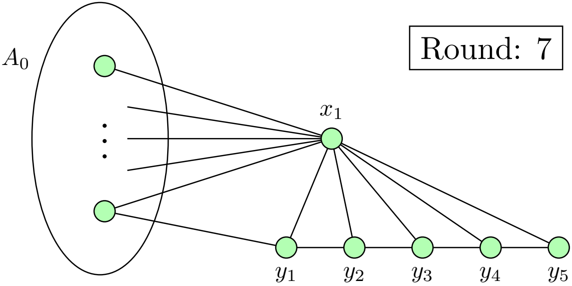

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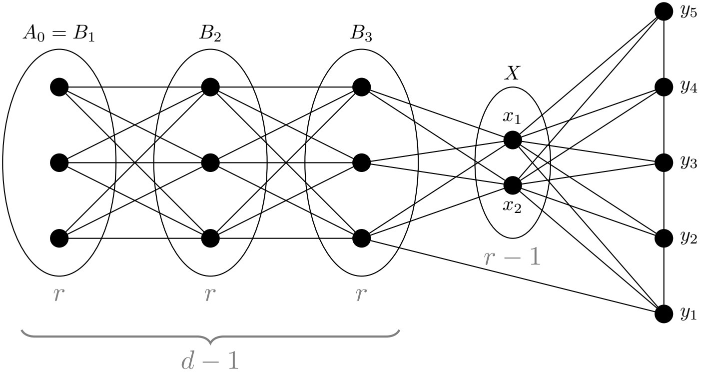
Question: What is the maximum number of rounds until percolation?
For arbitrary r and diameter d. (Example: $r = 3$, $d = 4$.)



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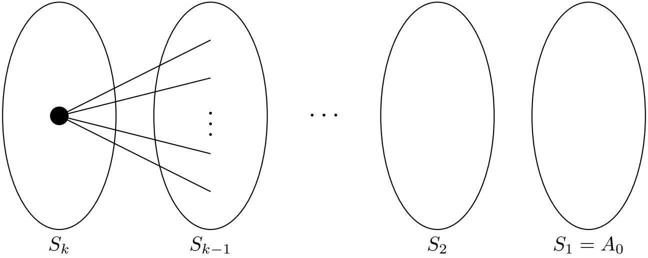

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Thank you!

arXiv:2309.13138

Number of Rounds

Theorem (Ibrahim, LaFayette, McCall '23)

Let G be a connected graph with diameter d. Suppose G contains a set of vertices, A_0, which percolates with threshold r in k rounds and $|A_0| \leq 2r - 1$. Furthermore, assume that every vertex in A_0 infects some vertex in round 2, i.e., every vertex in A_0 is adjacent to at least one vertex in round 2. Then $k \geq \lceil d/2 \rceil + 1$ and this bound is sharp.

Theorem (Ibrahim, LaFayette, McCall '23)

Let G be a connected graph with a set of vertices A_0, which percolates in k rounds with percolation threshold r. If $|A_0| = r$, then $k \geq rad(G) + 1$ and this bound is sharp.